École Normale Supérieure Paris-Saclay

master 1 jacques hadamard – rapport de stage de recherche

Cauchy horizons, from regularity to symmetry

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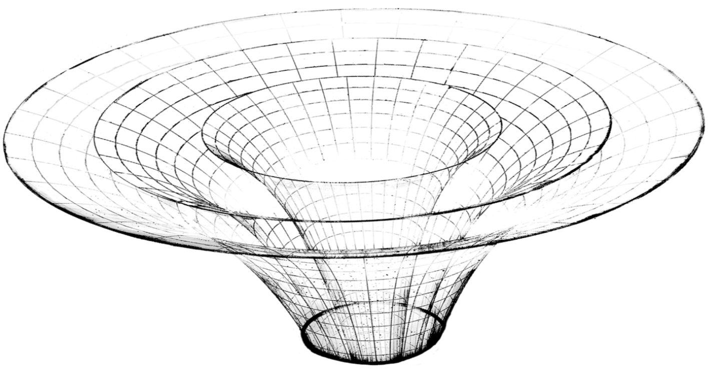

Work done in Florence, April – July 2021.

Contexte

Ce rapport présente le travail que j'ai effectué lors de mon stage de recherche du M1 Jacques Hadamard de l'ENS Paris-Saclay du 19 avril au 24 juillet 2021. Mon encadrant était Ettore Minguzzi, et la structure d'accueil était le département de mathématiques de l'université de Florence. Pour mon stage, je souhaitais travailler sur un sujet mêlant géométrie et physique, donc j'ai contacté Ettore qui m'a proposé un sujet sur les horizons de Cauchy, ce qui m'a tout de suite intéressé. Les prérequis pour travailler dans ce domaine sont des notions relativité mathématiques et de géométrie semi-Riemannienne, donc avant le stage j'ai consolidé mes connaissances dans ce sujet, en lisant [HE73] et [ONe83].

Une fois arrivé à Florence, j'ai commencé par étudier l'article [RB20b] qui présentait un argument prometteur, le *ribbon argument*, qui montre un lien entre les différents générateurs d'un horizon de Cauchy. Étant de l'avis que l'article ne concluait pas sur le problème de manière satisfaisante, nous avons commencé une preuve de la normalisation de la *surface gravity* en modifiant le *ribbon argument* et son utilisation. Mi-juin, nous avons achevé la preuve de l'incomplétude des générateurs, puis la preuve que la longueur affine est lisse, les conséquences géométriques, ainsi que le cas d'un horizon dégénéré ont occupé le mois restant.

Le département de mathématiques de l'université de Florence est réparti sur deux sites. La partie Physique Mathématiques, où était mon bureau, est située dans un ancien monastère sur une colline au dessus de Florence, c'est un lieu de travail très agréable.

Acknowledgements

I would like to sincerely thank Ettore Minguzzi for giving me the opportunity to work on this truly fascinating topic, and for his dedication to overcome the mathematical issues that we faced. I would like to thank him as well for our interesting conversations, and for inviting me to his seminar *Singularity theorems, causality, and all that.*

Grazie a Chiara, Carlotta e Priscilla for all the laughs we had during these three months, our trips in and out of Florence, the « compliments » on my carbonara, and everything else !

Enfin, merci à Frédéric Pascal pour son implication dans notre formation.

Abstract

We prove that in a smooth spacetime satisfying the *dominant energy condition*, the null generators of a future non-degenerate compact Cauchy horizon are all future-incomplete. We then prove the smoothness of the affine length function. A corollary of this result is the existence of a smooth nowhere-zero null vector field on the horizon that normalises the surface gravity to a non-zero constant. We mention a few consequences on the geometry and symmetry of the horizon, and we show that the horizon is in the closure of the chronology violating set.

Résumé

Nous prouvons que dans un espace-temps vérifiant la *dominant energy condition*, les générateurs d'un horizon de Cauchy futur compact non-dégénéré sont tous incomplets vers le futur. Nous prouvons ensuite que la longueur affine est lisse. Un corollaire de ce résultat est l'existence d'un champs de vecteur de type lumière qui normalise la *surface gravity* à une constante non nulle. Nous évoquons quelques conséquences sur la géométrie et symétrie de l'horizon, et nous montrons que l'horizon est dans la fermeture du *chronology violating set*.

mathematics subject classification 2020 – 53C50, 53C25, 53Z05, 83C75.

Contents

1 Introduction

One half of the 2020 Physics Nobel Prize was awarded to Roger Penrose "for the discovery that black hole formation is a robust prediction of the general theory of relativity". In his own words (seminar SCRI21), this description is wrong in the sense that his work, in particular his 1965 Singularity Theorem, predicts the robust formation of singularities, and not of black holes. The fact that singularities are necessarily hidden inside black holes is the object of the 1969 *Penrose Cosmic censorship hypothesis*, and has yet to be confirmed.

However, some progress [MI83; MI18; PR18; Pet19; RB20b] has recently been made in that direction. To understand it, it is useful to reformulate the Cosmic censorship hypothesis in the following way : *the maximal Cauchy development of generic compact or asymptotically flat initial data is locally inextendible as a regular Lorentzian manifold.* Though complicated at first sight, we will see later that this formulation highlights the facts that the existence of a naked singularity (*i.e.* not inside a black hole) implies the existence of a chronology violating set inside the spacetime, hidden behind a *Cauchy horizon.* Thus, to make a step towards the proof of a Cosmic censorship hypothesis, one can focus on Cauchy horizons, and try to find properties that they satisfy, in order to discriminate their existence or not.

This report presents a modest such step : the proof that Cauchy horizons present some kind of symmetry, when assumed to be compact, non future-degenerate, and in a smooth spacetime satisfying the dominant energy condition. Our result actually holds for more general compact connected smooth totally geodesic null hypersurfaces.

Some preliminary definitions and results are presented in Section 2 (see [ONe83], [HE73] or [Wal84] for reference). The objective of Section 3 is to clarify what was stated above, about the alleged non-genericity of Cauchy horizons, and to present the context (the Isenberg-Moncrief conjecture) of the results of this report. Section 4 presents the different energy conditions and the fundamental null-closeness property of a special one-form on the horizon, called the connection form. This property is then used in Section 5, where we prove the main result of this report (the vocabulary will be introduced throughout the sections) :

Theorem 1.1

Let $\mathcal{H}$ be a non-degenerate compact Cauchy horizon in a smooth spacetime satisfying the dominant energy condition. Then $\mathcal{H}$ admits a smooth nowhere-zero null vector field h that normalises the surface gravity to -1, i.e. such that

$$\nabla_h h = -h \ .$$

This results shows that the property of *normalisation to a non-zero constant of the surface gravity* (*i.e.* the existence of n such that $\nabla_n n = \kappa n$ with a constant $\kappa \neq 0$), that is satisfied for all the known examples of compact Cauchy horizons, holds for any compact Cauchy horizon under the reasonable *dominant energy condition.* Note that a version of the Cosmic censorship was proved in [Min15, Theorem 25], for which Theorem 1.1 is a kind of critical case (in vacuum, or when the weakened stable dominant energy condition doesn't hold), under which compact Cauchy horizons may exist, but satisfy some sort of rigidity.

Let us mention that this result was proved, in the vacuum case, by Martín Reiris and Ignacio Bustamante in 2020 in [RB20b]. A few doubts about this work led us to try to prove ourselves the result, removing the vacuum hypothesis, in a different (and coordinate-free) way. Along the way, we introduced useful tools on Cauchy horizons that allowed us to prove new results, notably on the affine length function, on the curvature tensor, and on the globalness of the behaviors of the generators. In the end, we think that the proof in [RB20b] is correct, though complicated.

We also wrote the results presented in this report in the form of a submitted article *Surface gravity of compact non-degenerate horizons under the dominant energy condition* [GM21].

2 Mathematical preliminaries

2.1 Null hypersurfaces

The notions of *spacetime, null/spacelike/lightlike/timelike vectors* are presented in Appendix B.1. Let $(\mathcal{M}, g)$ be a spacetime of dimension $n+1$. As presented in Appendix B.1, a vector $X \in T\mathcal{M}$ is *null* if $g(X, X) = 0$. A *null hypersurface* $\mathcal{H} \subseteq \mathcal{M}$ is a differential hypersurface $\mathcal{H}$ of $\mathcal{M}$ (that is, a differential submanifold of dimension n) such that for every $p \in \mathcal{H}$, the normal vector of $T_p\mathcal{H}$ in $T_p\mathcal{M}$ is null.

Note that this normal vector is well-defined up to scalar multiplication at every point p of $\mathcal{H}$. Indeed, as shown in [ONe83, p. 49],

$$\dim T_p\mathcal{H} + \dim T_p\mathcal{H}^{\perp g} = n + 1$$

so $\dim T_p\mathcal{H}^{\perp g} = 1$ and one can declare that $\mathcal{H}$ is a null hypersurface if any non-zero normal vector to $T_p\mathcal{H}$ is null. By definition, the restriction of the metric g to the null hypersurface $\mathcal{H}$ is degenerate, as the normal null vector must be in $T_p\mathcal{H}$.

For $p \in \mathcal{M}$, denote by $\mathcal{N}_p$ the null cone of $T_p\mathcal{M}$. A classic fundamental property of null hypersurfaces is as follows (a proof is presented below for completeness) :

Proposition 2.1

Let $\mathcal{H} \subseteq \mathcal{M}$ be a null hypersurface. For every $p \in \mathcal{H}$, $T_p\mathcal{H} \cap \mathcal{N}_p$ is a one-dimensional linear subspace of $T_p\mathcal{M}$.

Proof. Let $K \in \mathcal{N}_p$ such that $T_p\mathcal{H}^{\perp g} = \mathbb{R}K$. We need to show that if $v \in T_p\mathcal{H} \cap \mathcal{N}_p$, K and v are linearly dependent. K and v are orthogonal null vectors, as $v \in T_p\mathcal{H} = (\mathbb{R}K)^{\perp g}$. Thus, it is enough to show that if a null vector u and a causal vector v of the Minkowski space $V = \mathbb{R}^{n+1}$, with the metric $g = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + ... + (\mathrm{d}x^n)^2$, are orthogonal, then they are linearly dependent. Let $e_0 = (1, 0, ..., 0) \in V$. As $e_0^{\perp g} = \{0\} \times \mathbb{R}^n$, we have $V = \mathbb{R}e_0 + e_0^{\perp g}$. Let us write in this decomposition $u = u_0 + \nu e_0$ and $v = v_0 + \mu e_0$ with $u_0, v_0 \in e_0^{\perp g}$. Then,

$$\begin{aligned} 0 = g(u, u) &= g(u_0 + \nu e_0, u_0 + \nu e_0) \\ &= g(u_0, u_0) + 2\nu g(u_0, e_0) + \nu g(e_0, e_0) \\ &= g(u_0, u_0) - \nu^2 \end{aligned}$$

$$\begin{aligned} 0 \geq g(v, v) &= g(v_0 + \mu e_0, v_0 + \mu e_0) \\ &= g(v_0, v_0) + 2\mu g(v_0, e_0) + \mu g(e_0, e_0) \\ &= g(v_0, v_0) - \mu^2 \end{aligned}$$

$$\begin{aligned} 0 = g(u, v) &= g(u_0 + \nu e_0, v_0 + \mu e_0) \\ &= g(u_0, v_0) - \mu\nu \end{aligned}$$

So that $g(\mu u_0 - \nu v_0, \mu u_0 - \nu v_0) \leq \mu^2\nu^2 + \mu^2\nu^2 - 2\mu\nu g(u_0, v_0) = 0$. But $\mu u_0 - \nu v_0 \in e_0^{\perp g} = \{0\} \times \mathbb{R}^n$ which is a spacelike subspace of V, so $\mu u_0 - \nu v_0 = 0$, which shows that $\mu u - \nu v = 0$ *i.e.* that u and v are linearly dependent. □

This property applied at every point also shows that the set of null nowhere-zero vector fields tangent to $\mathcal{H}$ is a $\mathcal{F}(\mathcal{H}, \mathbb{R}_+^*)$-module of dimension 1. The following proposition will allow us to freely consider global smooth normal null vector fields on null hypersurfaces.

Proposition 2.2

Let $\mathcal{H} \subseteq \mathcal{M}$ be a null hypersurface of a spacetime $(\mathcal{M}, g)$. There is a smooth nowhere-zero null vector field Z on $\mathcal{H}$ tangent to $\mathcal{H}$.

As introduced in Appendix B.1, by definition, a spacetime is time-oriented, thus contains a global timelike smooth vector field which can be chosen to be future directed (recall that a vector $T \in T\mathcal{M}$ is *timelike* if $g(T,T) < 0$).

Proof. Let T be a global timelike smooth vector field on $\mathcal{M}$. T must be transverse to the null hypersurface $\mathcal{H}$. Indeed, if $T\mathcal{H}$ contained a timelike vector, as shown in [ONe83, Lemma 5.27], it would be a timelike subspace of $T\mathcal{M}$, which is absurd as the metric is degenerate on $\mathcal{H}$. Thus $T \notin T\mathcal{H}$, and as $T\mathcal{H}$ is a hyperplane, $\mathbb{R}T \oplus T\mathcal{H} = T\mathcal{M}$.

Let β be the one-form on $\mathcal{M}$ such that $\beta(T) = 1$ and such that $\beta(X) = 0$ for every X tangent to $\mathcal{H}$. Now, let $Z = \beta^\sharp|_\mathcal{H}$ be the vector field defined on $\mathcal{H}$ metrically equivalent to β, *i.e.* such that $g(Z, \cdot\,) = \beta$ on $\mathcal{H}$. It is always possible to find this Z by defining the coordinates

$$Z^\mu = g^{\mu\nu}\beta_\nu$$

in some coordinate basis $(\partial_0, \dots, \partial_n)$ of $T_p\mathcal{M}$. Z is clearly smooth by the preceding formula, as β is smooth. Z is tangent to $\mathcal{H}$ because if $X \in T_p\mathcal{H}$ is non-zero such that $\mathbb{R}X = T_p\mathcal{H}^\perp$, we know that X must be null and hence tangent to $\mathcal{H}$, thus

$$g(Z(p), X) = \beta_p(X) = 0$$

so $Z(p) \in X^\perp = T_p\mathcal{H}$. Z is null because $g(Z,Z) = \beta(Z) = 0$ as $Z \in \Gamma\mathcal{H}$. Finally, V is nowhere zero because if $Z(p) = 0$, $1 = \beta_p(T(p)) = 0$ which is a contradiction. □

2.2 Cauchy horizons

Most of the vocabulary used in this section is defined in Appendix B.2. We introduce here the main object on which we will work : Cauchy horizons. Let $(\mathcal{M}, g)$ be a spacetime, and let $S \subseteq \mathcal{M}$. The *future* and *past Cauchy developments of* S are defined respectively as :

$$\mathcal{D}^+(S) = \left\{ p \in \mathcal{M} \;\middle|\; \text{every past inext. causal curve through } p \text{ intersects } S \right\}$$
$$\mathcal{D}^-(S) = \left\{ p \in \mathcal{M} \;\middle|\; \text{every future inext. causal curve through } p \text{ intersects } S \right\}.$$

Thus, the future Cauchy development of S is the set of points that are causally completely determined by S, *i.e.* if $q \in \mathcal{D}^+(S)$ is in the causal future of $p \in \mathcal{M}$, then p is in the causal future of S or the opposite.

We then define the *future* and *past Cauchy horizon of* S respectively as :

$$\mathcal{H}^+(S) = \overline{\mathcal{D}^+(S)} \backslash I^-(\mathcal{D}^+(S))$$
$$\mathcal{H}^-(S) = \overline{\mathcal{D}^-(S)} \backslash I^+(\mathcal{D}^-(S)) .$$

Recall that for $P \subseteq \mathcal{M}$, $I^+(P)$ (resp. $I^-(P)$) is the chronological future (resp. past) of P, which consists of the set of point $q \in \mathcal{M}$ such that q is the end (resp. beginning) of a timelike curve starting (resp. ending) in P, as introduced in Annex B.2.

Thus, the future Cauchy horizon of S consists in the "future boundary" of $\mathcal{D}^+(S)$, *i.e.* consists in the set of points in the boundary of $\mathcal{D}^+(S)$ which are in the chronological future of no points of $\mathcal{D}^+(S)$. In some way, the future Cauchy horizon of S is the boundary beyond which spacetime cannot be predicted with the only knowledge of S. This interpretation is illustred by the following property, proved in [Min19, p. 89] :

Proposition 2.3

Let $S \subseteq \mathcal{M}$ be closed and achronal. Then $\mathcal{H}^+(S) = \varnothing$ if and only if

$$S \cup I^+(S) = \left\{ p \in \mathcal{M} \,\middle|\, \textit{every past inext. timelike curve through } p \textit{ intersects } S \right\} .$$

Thus, the future Cauchy horizon of a closed achronal set S is empty if and only if the chronological future of S can be predicted by the knowledge of S. From this fact, we can begin to believe that the presence of Cauchy horizons is not physically acceptable, as it goes against the concepts of determinism and predictability. This will be discussed in Section 3.

As shown in [CI94], Cauchy horizons can be arbitrarily complicated sets. However, assuming compactness of a Cauchy horizon $\mathcal{H}$ gives a lot of information.

Totally geodesic submanifolds

Denote ∇ the Levi-Civita connection of a spacetime $(\mathcal{M}, g)$. Given a general submanifold $\mathcal{W} \subseteq \mathcal{M}$ and $X, Y \in \Gamma\mathcal{W}$, as shown in [ONe83, p. 98], there are local extensions of X, Y to vector fields on open coordinate sets of $\mathcal{M}$, that we still denote X, Y. We can then define the covariant derivative $\overline{\nabla}$ restricted to $\mathcal{W}$ as :

$$\overline{\nabla}_X Y = (\nabla_X Y)\Big|_{\mathcal{W}} .$$

It can be shown, for example in [ONe83, Lemma 4.1] that $\overline{\nabla}_X Y$ only depends on the value of X and Y on $\mathcal{W}$, and not on the coordinate extension. However, this definition doesn't define a connection on $\mathcal{W}$ in general, as $\overline{\nabla}_X Y$ might not be tangent to $\mathcal{W}$. When it is the case, we say that $\mathcal{W}$ is a *totally geodesic submanifold of* $\mathcal{M}$, and then $\overline{\nabla}$, that we relabel ∇, is a connection on $\mathcal{W}$ that inherits properties of the original connection ∇. For example if $X, Y, Z \in \Gamma\mathcal{W}$,

$$\nabla_Z[g(X, Y)] = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) .$$

Indeed, this equation is just the restriction to $\mathcal{W}$ of the same equation on $\mathcal{M}$ replacing X, Y, Z by their extensions.

The following property, which follows from [Min15, Theorem 18] is fundamental.

Theorem 2.1

Let S be a connected partial Cauchy hypersurface of a smooth spacetime $(\mathcal{M}, g)$ on which the null energy condition holds, and suppose that its future Cauchy horizon $\mathcal{H}^+(S)$ is compact. Then $\mathcal{H}^+(S)$ is a smooth totally geodesic null hypersurface of $(\mathcal{M}, g)$, ruled by null geodesics.

The null energy condition will be introduced in Section 4. As introduced in Appendix B.2, a partial Cauchy hypersurface is an acausal and edgeless hypersurface of $\mathcal{M}$. The last statement of Theorem 2.1 means that for every $p \in \mathcal{H}^+(S)$, there is a unique null geodesic through p contained in $\mathcal{H}^+(S)$. These null geodesics will be called the *generators* or the *null generators* of the horizon. From now on, we will call simply *compact Cauchy horizon* any connected compact future Cauchy horizon $\mathcal{H}^+(S)$ where S is as in Theorem 2.1.

Proof of Theorem 2.1. The main steps are presented here. Denote $\mathcal{H} = \mathcal{H}^+(S)$.

Topological hypersurface : As S is acausal, $I^+(S) \cap I^-(S) = \varnothing$. If we denote the past set $P = D^+(S) \cup I^-(S)$, we have by Proposition B.4 that, as $I^+(S)$ and $I^-(S)$ are open,

$$\mathcal{H} = I^+(S) \cap \partial D^+(S) = I^+(S) \cap \partial P .$$

Moreover, by Corollary B.1, as P is a past set, ∂P is a topological hypersurface of $\mathcal{M}$. Thus, as $I^+(S)$ is open, $\mathcal{H}$ is a topological hypersurface of $\mathcal{M}$.

Smooth hypersurface : This is the most difficult step of the proof, which uses compactness and the null energy condition, and can be found in [Min15, Theorem 18]. This theorem also shows that, denoting n the normal to $\mathcal{H}$, the Weingarten map

$$b: \begin{array}{ccc} T\mathcal{H}/n & \longrightarrow & T\mathcal{H}/n \\ \overline{X} & \longmapsto & \overline{\nabla_X n} \end{array}$$

is identically zero, $i.e.$ for every $X \in T\mathcal{H}$, $b(X)$ is proportional to n.

Ruled by null geodesics : This well-know fact directly comes from the definition of the Cauchy horizon and can be found for example in [HE73, p. 204].

Null hypersurface : This comes from the preceding fact and the achronality of $\mathcal{H}$. Note that $\mathcal{H} = \overline{D^+(S)} \backslash I^-(D^+(S))$ is indeed achronal, because if $\gamma : p \longrightarrow q$ is a timelike curve connecting two points $p, q \in \mathcal{H}$ then by openness of the chronological relation I, $p \in I^-(\overline{D^+(S)}) \subset I^-(D^+(S))$ which is absurd as $p \in \mathcal{H}$. Now, let $w \in T_p\mathcal{M}$ be a non-zero normal vector to $T_p\mathcal{H}$. If w was spacelike, $T_p\mathcal{M}$ would contain some timelike vector that would be the derivative of a locally timelike curve traced on $\mathcal{H}$, which is absurd by achronality. Thus w is causal and orthogonal to the initial velocity v of the null generator leaving p, thus is proportional to v by the demonstration of Proposition 2.1, hence is null.

Totally geodesic : Let $X, Y \in \Gamma\mathcal{H}$. We need to show that $\nabla_X Y$ is tangent to $\mathcal{H}$ at any $p \in \mathcal{H}$. Let $N \in T_p\mathcal{M}$ be a null vector independant of n, the null normal to $T_p\mathcal{H}$. As $g(N, n) \neq 0$ by the demonstration of Proposition 2.1, we can rescale N to get $g(N, n) = -1$. Now, the coordinate of $\nabla_X Y$ on N in the splitting $T_p\mathcal{M} = T_p\mathcal{H} + \mathbb{R}N$ is

$$-g(n, \nabla_X Y) = g(\nabla_X n, Y) = 0$$

because $g(\nabla_X n, Y) + g(n, \nabla_X Y) = \nabla_X[g(n, Y)] = 0$ and because $\nabla_X n = b(X)$ is proportional to n which is orthogonal to Y. Hence the result. □

2.3 Existence of a smooth horizontal distribution

Let $\mathcal{H}$ be a compact Cauchy horizon. Let $\mathcal{N} \longrightarrow \mathcal{H}$ be the null bundle of $\mathcal{H}$. As proved in Proposition 2.2, there is a smooth null nowhere-zero vector field $Z \in \Gamma\mathcal{H}$, and by Proposition 2.1, we have $\mathcal{N} = \mathbb{R}Z$. We call *horizontal distribution* any smooth subbundle $H \longrightarrow \mathcal{H}$ of the tangent bundle $T\mathcal{H}$ supplementary to $\mathcal{N}$ in $T\mathcal{H}$, $i.e.$ such that $\mathcal{N} \oplus H = T\mathcal{H}$, where

$$\mathcal{N} \oplus H = \bigsqcup_{p \in \mathcal{H}} \{p\} \times (\mathcal{N}_p \oplus H_p) \ .$$

The aim of this section is to show the existence of such a horizontal distribution H. Denote V a global smooth timelike vector field on $\mathcal{M}$.

Proposition 2.4

The bundle $H := V^\perp \cap T\mathcal{H} := \bigsqcup_{p \in \mathcal{H}} \{p\} \times \left(V(p)^\perp \cap T_p\mathcal{H}\right) \longrightarrow \mathcal{H}$ *is a horizontal distribution.*

Proof. The smoothness of H follows from the smoothness of V. Indeed, $V^\perp = \ker V^\flat$ is a smooth subbundle by Proposition A.4. $T\mathcal{H} \longrightarrow \mathcal{H}$ is also a smooth subbundle because $\mathcal{H}$ is a smooth submanifold of $\mathcal{M}$. Moreover, as V is nowhere zero, and as $T_p\mathcal{H}$ is a hyperplane of $T_p\mathcal{M}$ that is not equal to $V^\perp$ because $\mathcal{H}$ is a null hypersurface (the orthogonal of a timelike vector is spacelike as shown in [ONe83, Lemma 5.26]), $\dim(V(p)^\perp \cap T_p\mathcal{H}) = n - 1$ is independant of $p \in \mathcal{H}$. Thus H is smooth by Proposition A.5.

We also need to check that $\mathcal{N} \oplus H = T\mathcal{H}$. First, notice that $H_p = V(p)^\perp \cap T_p\mathcal{H}$ is a hyperplane of $T_p\mathcal{H}$, as $V(p)^\perp$ is a hyperplane of $T_p\mathcal{M}$ that doesn't contain $T_p\mathcal{H}$ otherwise $V(p) \in Z(p)^\perp$. This is absurd because V is timelike and, denoting $\mathscr{C}$ the causal cone in p, we saw in Section 2.1, $Z(p)^\perp \cap \mathscr{C} = \mathbb{R}Z(p)$. Thus, H_p is a hyperplane of $T_p\mathcal{M}$ and we only need to check that $Z \notin H_p$, which follows again from the fact that Z and V are not orthogonal. □

We will use this horizontal distribution H to have a global direction transverse to the null direction of $\mathcal{H}$. This will allow us to use the compactness of $\mathcal{H}$ to bound quantities defined on horizontal curves, which will be fundamental in Section 5 (see for example Lemma 5.3). Notice that a horizontal distribution H is necessarily spacelike, becauce $T\mathcal{H}$ doesn't contain any timelike vector because its normal is null but the orthogonal of a timelike vector is spacelike.

2.4 Horizontal geometry

This section presents some results of sub-Riemannian geometry. The proofs are included in appendix for completeness. Let $\mathcal{H}$ be a compact Cauchy horizon, and let H be a horizontal distribution. Denote

$$\pi : T\mathcal{H} = H \oplus \mathcal{N} \longrightarrow H$$

the canonical projection. A vector field $Y \in \Gamma\mathcal{H}$ is said to be horizontal if for every p, $Y(p) \in H_p$. We say that a curve is horizontal if its velocity is horizontal at every point.

We also say that a curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathcal{H}$ is a *horizontal geodesic* if for every s,

$$\gamma'(s) \in H_{\gamma(s)} \quad \text{and} \quad \pi\left(\nabla_{\gamma'}\gamma'(s)\right) = 0\ .$$

The following proposition will allow us to use horizontal geodesics the same way we use usual geodesics. Actually, we see in the proof that in order to show the local existence and uniqueness of horizontal geodesics, we can use the similar well-known result for usual geodesics in Riemannian manifolds.

Proposition 2.5

For every $p \in \mathcal{H}$ and $v \in H_p$, there is a unique horizontal geodesic $\gamma : \mathbb{R} \longrightarrow \mathcal{H}$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

The proof of Proposition 2.5 is the object of Appendix A.3. We can then define the *horizontal exponential* map

$$\begin{array}{rccc} \overline{\exp} : & H & \longrightarrow & \mathcal{H} \\ & (p,v) & \longmapsto & \overline{\gamma}_{p,v}(1) \end{array}$$

where $\overline{\gamma}_{p,v}$ is the horizontal geodesic defined on $\mathbb{R}$ such that $\overline{\gamma}_{p,v}(0) = p$ and $\overline{\gamma}'_{p,v}(0) = v$. The following property of the usual exponential map is still true for the horizontal exponential map :

Proposition 2.6

Let $p \in \mathcal{H}$ and $v \in H_p$. Denote by $\overline{\exp}_p$ the map $\overline{\exp}(p,\cdot) : H_p \longrightarrow \mathcal{H}$. Then $\mathrm{d}(\overline{\exp}_p)_0(v) = v$.

Proof. First, notice that $\overline{\gamma}_{p,v}(s) = \overline{\gamma}_{p,v}(s.\cdot 1) = \overline{\exp}(p, sv)$ because $t \longmapsto \overline{\gamma}_{p,v}(st)$ is the curve horizontal leaving p at velocity sv. If $\alpha(s) = sv$, we have $\alpha(0) = 0$, $\alpha'(0) = v$, thus

$$\mathrm{d}(\overline{\exp}_p)_0(v) = \mathrm{d}(\overline{\exp}_p)_{\alpha(0)}(\alpha'(0)) = \frac{\mathrm{d}}{\mathrm{d}s}(\overline{\exp}(p, \alpha(s))|_{s=0} = \overline{\gamma}'_{p,v}(0) = v\ .$$

□

3 The genericity of Cauchy horizons

This section provides the context of the results proved in the following sections, and their links with famous problems of General Relativity. The definition of a vacuum spacetime is presented in Section 4. We denote $\mathcal{L}$ the Lie derivative and ∇ the Levi-Civita connection.

3.1 Cosmic censorship and the Isenberg-Moncrief conjecture

As we began to see in Section 2.2, the occurrence of Cauchy horizons goes against the concept of determinism and predictability. For this reason, and other technical aspects, the particular case of spacetimes with Cauchy horizons is often put aside in General Relativity theorems. To justify this, Roger Penrose formulated in 1969 the *Cosmic censorship hypothesis*, which states (in the strong case) that the maximal Cauchy development of generic compact or asymptotically flat initial data is locally inextendible as a regular Lorentzian manifold. In other words, it states that there is a region whose predictability set is the whole inextendible spacetime.

This is typically not the case in the presence of Cauchy horizons where predictability breaks down, hence the series of articles [MI83; MI18; PR18; Pet19; RB20b] with the objective of discriminating the existence of Cauchy horizons, under physical assumptions on the spacetime.

A *Killing horizon* of a spacetime $(\mathcal{M}, g)$ is a null hypersurface $\mathcal{H}$ defined by the vanishing of the $g-$norm of a Killing vector field defined on a neighbourhood of $\mathcal{H}$ $i.e.$ there is an open set $\mathcal{U} \supseteq \mathcal{H}$ and a Killing vector field K (such that $\mathcal{L}_K g = 0$, see Appendix A) on $\mathcal{U}$ such that

$$\mathcal{H} = \left\{ p \in \mathcal{U} \,\middle|\, g(K(p), K(p)) = 0 \right\} .$$

In accordance with the non-genericity of Cauchy horizons in the compact case, Jim Isenberg and Vincent Moncrief conjectured in their article *Symmetries of cosmological Cauchy horizons* (1983) [MI83] that compact Cauchy horizons of smooth vacuum spacetimes are Killing horizons. It is a known fact that spacetimes with Killing fields are non-generic (for example, see [MM15] which shows that the set of spacetimes with no Killing fields is an open and dense subset of the set of all spacetimes, for a suitable topology). Thus, if proved, this Isenberg-Moncrief conjecture goes in favor of the Penrose Cosmic censorship hypothesis, by showing that vacuum spacetimes with compact Cauchy horizons admit a Killing symmetry, which is a non-generic property.

3.2 The proof strategy

We say that *the surface gravity of a compact Cauchy horizon $\mathcal{H}$ of a spacetime $(\mathcal{M}, g)$ can be normalized to a non zero constant* if there is a smooth lightlike nowhere-zero vector field K on $\mathcal{H}$, tangent to $\mathcal{H}$, such that $\nabla_K K = \kappa K$, with κ a non-zero constant. Note that we can suppose $\kappa = -1$. Indeed, rescaling $V := -K/\kappa$, we get

$$\nabla_V V = \frac{1}{\kappa^2} \nabla_K K = \frac{1}{\kappa} K = -V .$$

In [PR18] and [Pet19], Oliver Lindblad Petersen and István Rácz proved[1] that in a vacuum spacetime, a compact Cauchy horizon whose surface gravity can be normalised to a non-zero constant is a Killing horizon. Let us call *pre-Killing* field any smooth lightlike nowhere-zero vector field K on $\mathcal{H}$, tangent to $\mathcal{H}$, such that

$$\nabla_K K = -K . \tag{3.1}$$

Thus, to demonstrate the Isenberg-Moncrief conjecture in the class of smooth vacuum spacetimes, it is enough to show any compact Cauchy horizon has a pre-Killing field. However, in [MI83],

[1] it is worth to mention that we pointed out to them a mistake spotted in [PR18, Lemma 2.8], that was fixed and will be addressed in the next version of their article.

Isenberg and Moncrief found examples of null hypersurfaces for which this result is not true. These examples all have the specificity that their null generators are future-complete. A compact Cauchy horizon is said to be *non-degenerate* if at least one of its null generators is future-incomplete. Though it is unclear if these "degenerate" null hypersurfaces in [MI83] can be realized as Cauchy horizons, we can still state a more precise version of the Isenberg-Moncrief conjecture as follows :

Isenberg-Moncrief conjecture (weaker version)

The surface gravity of a non-degenerate connected compact Cauchy horizon can be normalized to a non-zero constant.

Even though the conjecture is stated for vacuum spacetimes, we will see that this result is true for Cauchy horizons of spacetimes satisfying a weaker energy condition, introduced in Section 4. The proof of this result is the object of Section 5. Before all this though, we will discuss a way to reduce this problem into two different steps.

3.3 The candidate vector field

First, let us find a necessary condition for K to be a pre-Killing vector on $\mathcal{H}$. Suppose for a moment that there is a pre-Killing vector field K on $\mathcal{H}$. Let $p \in \mathcal{H}$ and $\gamma : I = [0, L) \longrightarrow \mathcal{H}$ be the unique future-inextensible geodesic such that $\gamma(0) = p$, $\gamma'(0) = K(p)$ (recall that $\mathcal{H}$ is totally geodesic). For every $s \in I$, $\gamma'(s)$ is null (as the causal character of a geodesic is constant) and $\gamma(s) \in \mathcal{H}$ so $\gamma'(s) \in T_{\gamma(s)}\mathcal{H} \cap \mathcal{N}_{\gamma(s)}$. By Proposition 2.1, as K is nowhere-zero lightlike tangent to $\mathcal{H}$, there is a function $f : I \longrightarrow \mathbb{R}\backslash\{0\}$ such that for every $s \in I$,

$$\gamma'(s) = f(s)K(\gamma(s)) \; . \tag{3.2}$$

It is easy to see that f is smooth. Moreover, as K satisfies Equation (3.1), and as γ is a geodesic, we have

$$\begin{aligned} 0 &= \nabla_{\gamma'(s)}\gamma'(s) \\ &= \nabla_{\gamma'(s)}(f(s)K(\gamma(s))) \\ &= f'(s)K(\gamma(s)) + f(s)\nabla_{f(s)K(\gamma(s))}K(\gamma(s)) \\ &= f'(s)K(\gamma(s)) + f(s)^2 \left(\nabla_K K\right)(\gamma(s)) \\ &= (f'(s) - f(s)^2)K(\gamma(s)) \end{aligned}$$

so f satisfies $f' = f^2$. As f is never zero, we have $(1/f)' = -f'/f^2 = -1$, and after integration and the fact that $f(0) = 1$ as $\gamma'(0) = K(p)$, we find

$$f(s) = \frac{1}{1-s}$$

on $I = [0, L)$, which shows that $L \leq 1$. If we had $L < 1$, by compactness we could find a converging sequence $s_n \longrightarrow L$, $\gamma(s_n) \longrightarrow q$, which implies $\gamma'(s_n) \longrightarrow K(q)/(1-L)$. This is absurd as we can now extend γ with the null geodesic leaving q with velocity $K(q)/(1-L)$, in contradiction with the inextensibility of γ. This shows that $L = 1$, and L is by definition the affine length of the geodesic γ (see Appendix A).

Thus, $K(p)$ *is the initial velocity of the lightlike geodesic starting at* p *with affine length 1.*

This fact was noticed in [RB20b], where was proposed the following proof strategy : if $\mathcal{H}$ is any non-degenerate compact Cauchy horizon, show that

(i) All the future null geodesics of $\mathcal{H}$ have finite affine length.

(ii) The candidate vector field K defined on $\mathcal{H}$ by
$K(p)$ is the initial velocity of the lightlike geodesic starting at p with affine length 1
is a pre-Killing field.

Note that if (i) is satisfied (and if K is well-defined, which we will show just after), K will automatically be lightlike and nowhere-zero, so we would only need to show that K is smooth and satisfies Equation (3.1).

Under the assumption that (i) is satisfied, let us precise the definition of the vector field K in (ii). Proposition 2.2 allows us to consider a global null nowhere-zero vector field Z on $\mathcal{H}$. For $p \in \mathcal{H}$, denote by $L(p)$ the affine length of the null geodesic leaving p at velocity $Z(p)$, and then define

$$K(p) := L(p)Z(p) \ .$$

Notice that this definition doesn't depend on the vector field Z chosen. Indeed, if V is another one, let $\lambda \in \mathbb{R}$ such that $V(p) = \lambda Z(p)$. Let $\gamma : [0, L_V(p)) \longrightarrow \mathcal{H}$ (resp. $\beta : [0, L_Z(p)) \longrightarrow \mathcal{H}$) be the geodesic leaving p at velocity $V(p)$ (resp. $Z(p)$). Necessarily, $\gamma(t) = \beta(\lambda t)$. Indeed, $t \longmapsto \beta(\lambda t)$ is clearly a geodesic leaving p in $t = 0$ with velocity $\lambda\beta'(0) = V(p)$. Thus, the inextendibility condition reads $\lambda L_V(p) = L_Z(p)$, which shows

$$L_V(p)V(p) = L_Z(p)Z(p) \ .$$

Moreover, $K(p)$ is the unique initial velocity for which the null geodesic starting at p has an affine length 1, as the calculation above shows that the affine length of the null geodesic starting at p with velocity $\lambda Z(p)$ is $L(p)/\lambda$.

Also note that to show that K is smooth, it is enough to show that the function L is smooth. Still under the assumption that (i) is satisfied, and supposing that K is smooth, let us show that (ii) is satisfied.
Let $p \in \mathcal{H}$ and let $\gamma : [0, 1) \longrightarrow \mathcal{H}$ be the inextensible null geodesic starting at p with velocity $K(p)$. By Proposition 2.1, there is a function f such that for every $s \in [0, 1)$,

$$\gamma'(s) = f(s)K(\gamma(s)) \ .$$

We necessarily have $f(s) = 1/(1 - s)$. Indeed, if $s_0 \in [0, 1)$, the null geodesic α starting at $\gamma(s_0)$ with velocity $\gamma'(s_0)$ is $\gamma|_{[s_0,1)}$, and has an affine length of $1 - s_0$. Thus, the null geodesic β starting at $\gamma(s_0)$ with velocity $(1 - s_0)\gamma'(s_0)$ clearly satisfies $\beta(t) = \alpha((1 - s_0)t)$, and thus has an affine length of $(1 - s_0)/(1 - s_0) = 1$. By uniqueness,

$$K(\gamma(s_0)) = (1 - s_0)\gamma'(s_0) \ .$$

We can now compute :

$$\begin{aligned}(\nabla_K K)(p) &= (\nabla_{\gamma'} K \circ \gamma)(0) \\ &= \nabla_{\gamma'}(1 - s)\gamma'(s)|_{s=0} \\ &= (1 - 1)\nabla_{\gamma'}\gamma'(0) - \gamma'(0) \\ &= -K(p)\end{aligned}$$

which shows that (ii) is satisfied. Thus, to prove the conjecture, one only needs to prove (i), and then check that the candidate vector field is smooth. In Section 5.2, we will show (i), and then show that the affine length function is smooth in Section 5.3 and Appendix C. In this objective, before that, we will study in Section 4 the fundamental properties of a special one-form, under energy assumtions on the spacetime.

4 Energy conditions and the connection form

Recall that on a totally geodesic hypersurface $\mathcal{H}$, one can restrict the spacetime connection ∇ to $\mathcal{H}$ to differentiate vector fields on $\mathcal{H}$ and get fields that are still tangent to $\mathcal{H}$, see Section 2.2. The fact that $T\mathcal{H} \cap \mathcal{N}$ is a one-dimensional vector bundle will allow us to define a particular one-form on $\mathcal{H}$, that will be related to curvature and to the affine length of null geodesics on $\mathcal{H}$.

Lemma 4.1

Let $Z \in \Gamma\mathcal{H}$ be a nowhere-zero null vector field on $\mathcal{H}$. For every vector field $Y \in \Gamma\mathcal{H}$, $\nabla_Y Z$ is a null vector field.

Proof. As Z is normal to $T\mathcal{H}$, we have $g(\nabla_Y Z, Z) = 0$. Thus, by property of the Levi-Civita connection (recall that the restriction of ∇ to $\mathcal{H}$ is defined by extending fields on $\mathcal{H}$ to a neighbourhood of $\mathcal{H}$), we have

$$\begin{aligned} 0 &= \nabla_Y[g(\nabla_Y Z, Z)] \\ &= g(\nabla_Y^2 Z, Z) + g(\nabla_Y Z, \nabla_Y Z) \\ &= g(\nabla_Y Z, \nabla_Y Z) \end{aligned}$$

which shows that $\nabla_Y Z$ is null. □

We then deduce the following proposition :

Proposition 4.1

For every nowhere-zero null vector field $Z \in \Gamma\mathcal{H}$, there is a one-form ω_Z on $\mathcal{H}$ such that for every $Y \in T\mathcal{H}$,

$$\nabla_Y Z = \omega_Z(Y) Z \; .$$

Proof. Let $Y \in \Gamma\mathcal{H}$. By Lemma 4.1 and Proposition 2.1, we get that at every point of $\mathcal{H}$, as Z is non-zero, $\nabla_Y Z$ is proportionnal to Z, hence the existence of a function $\omega_Z(Y) : \mathcal{H} \longrightarrow \mathbb{R}$ satisfying $\nabla_Y Z = \omega_Z(Y)Z$. Moreover, ω_Z is clearly $\mathcal{F}(\mathcal{H})$-linear by definition. Thus, to show that ω_Z is a one-form, we only need to check that for any Y, $\omega_Z(Y)$ is smooth. We can do this locally. Let $p \in \mathcal{H}$ and $\mathcal{V} \hookrightarrow \mathcal{M}$ containing p and a frame field $(e_0, \dots, e_n)$ on a $\mathcal{V}$, *i.e.* vector fields e_i such that at every point, (e_i) is a g-orthonormal basis of $T\mathcal{M}$ (see [ONe83, p. 84] for the existence). We can compute

$$Z = -g(Z, e_0)e_0 + \sum_{i=1}^{n} g(Z, e_i)e_i$$

$$\nabla_Y Z = -g(\nabla_Y Z, e_0)e_0 + \sum_{i=1}^{n} g(\nabla_Y Z, e_i)e_i \; .$$

Moreover, Z is null and non-zero, which shows that $-g(Z, e_0)$, the coordinate of Z on e_0, is non zero. Thus, we get on $\mathcal{V} \cap \mathcal{H}$ that

$$\omega_Z(Y) = \frac{g(\nabla_Y Z, e_0)}{g(Z, e_0)}$$

is smooth. □

We will see after this section how to link ω_Z with the affine length of null geodesics. Before that, in the case where $\mathcal{H}$ is a compact Cauchy horizon, and under a physical assumption on the spacetime $(\mathcal{M}, g)$, we will show a fundamental property of ω_Z, described in the following theorem. We denote $\mathrm{d}\omega$ the exterior derivative of a form ω. Notice that actually, we can define ω_Z for any non-necessarily smooth nowhere-zero null vector field Z. If Z is $\mathscr{C}^k$, ω_Z will be $\mathscr{C}^{k-1}$.

Theorem 4.1

Suppose that the spacetime $(\mathcal{M}, g)$ satisfies the dominant energy condition. Let $\mathcal{H} \subseteq \mathcal{M}$ be a compact Cauchy horizon. Let Z be a nowhere-zero smooth null vector field on $\mathcal{H}$. The one-form ω_Z on $\mathcal{H}$ defined in Proposition 4.1 is null-closed, that is, for every $Y \in T\mathcal{H}$,

$$\mathrm{d}\omega_Z(Z, Y) = 0 \ .$$

We will introduce the *dominant energy condition* just after. Note that this result implies that for any null vector field X tangent to $\mathcal{H}$, $\mathrm{d}\omega_Z(X, Y) = 0$. Also note that this result makes sense only if the compact Cauchy horizon $\mathcal{H}$ is a smooth null totally geodesic hypersurface. We know since Section 2.2 that it is true if the *null convergence condition* is satisfied. Thanksfully, we will see that it is implied by the dominant energy condition.

The conclusion of Theorem 4.1 was proved in gaussian coordinates by Vincent Moncrief and James Isenberg in [MI18, Section E] under the assumption that the spacetime is vacuum. We show here, in a simpler proof, that this result is still true under the weaker assumption that the spacetime satisfies the *dominant energy condition* (DEC), that we introduce now.

In accordance with the Einstein field equations, define the *stress-energy tensor* T such that

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \lambda g_{\mu\nu} = k T_{\mu\nu} \tag{4.1}$$

where R is the scalar curvature, $R_{\mu\nu}$, $g_{\mu\nu}$ are the coordinates of the Ricci tensor Ric and of the metric g in a local coordinate basis $(x^0, \dots, x^n)$, k is the Einstein gravitational constant $k = 8\pi G/c^4$, and λ is the cosmological constant (not assumed to be zero in this work). The energy conditions that we will mention are as follows :

(i). ***Vacuum spacetime :***
The stress-energy tensor of the spacetime $(\mathcal{M}, g)$ is zero.

(ii). ***Dominant energy condition :***
For every future-directed causal vector $X^\mu \partial_\mu$, the vector $T^\nu_\mu X^\mu \partial_\nu$ is causal past-directed.

(iii). ***Null energy condition :***
For every null vector V, $T(V, V) \geq 0$. Note that the Einstein field equations (4.1) show that it is equivalent to $\mathrm{Ric}(V, V) \geq 0$.

Recall that the raising of indices is defined by $T^\nu_\mu = g^{\nu\alpha} T_{\alpha\mu}$ where $g^{\nu\alpha}$ is the inverse matrix of $g_{\nu\alpha}$. Clearly, (i) $\implies$ (ii). We also have (ii) $\implies$ (iii) as $T(V, V) = T_{\mu\nu} V^\mu V^\nu = g_{\mu\alpha} T^\alpha_\nu V^\mu V^\nu = g(V, T^\nu_\mu V^\mu \partial_\nu) \geq 0$ because V and $T^\nu_\mu V^\mu \partial_\nu$ are causals in opposite time directions.

Physically, the null energy condition states that the energy density must be positive, the DEC states that mass-energy cannot travel faster than the speed of light, and the vacuum condition states that there is no matter-energy. Clearly, the DEC is more physically acceptable than the vacuum condition in general.

The function $\omega_Z(Z)$ is called *surface gravity* (associated to Z). Notice that the objective of Section 3.3 is to normalise the surface gravity to -1, *i.e.* find Z such that $\omega_Z(Z) = -1$. We will now prove Theorem 4.1. Let $\mathcal{H}$ and Z be like in the hypothesis of Theorem 4.1.

Lemma 4.2

Let $v \in \mathcal{N}$ be a null vector. If we denote by C the causal cone, we have

$$v^\perp \cap \mathsf{C} = \mathbb{R}v \ .$$

Proof. The inclusion $\supseteq$ is clear. Suppose that there is a causal vector $w \in v^\perp$ linearly independent of v. The demonstration of Proposition 2.1 shows that w is timelike. But then $v \in w^\perp$ which is a spacelike subspace, as shown in [ONe83, Lemma 5.26]. This is a contradiction as v is null. □

Proposition 4.2

Under the dominant energy condition, the one-form $\mathrm{Ric}(Z, \cdot\,)$ *is identically zero on* $\mathcal{H}$.

Proof. The important point, as shown in [Min15, Theorem 18], is that

$$\mathrm{Ric}(Z, Z) = 0\,. \tag{4.2}$$

Let U be the vector field defined on $\mathcal{H}$ metrically equivalent to the one-form $T(Z, \cdot)$, *i.e.* such that $g(U, \cdot\,) = T(Z, \cdot\,)$ (U is not necessarily tangent to $\mathcal{H}$, for the existence, see [ONe83, Prop. 3.10], or take the following calculation as a definition of U). Let us calculate the coordinates of U in some local coordinate basis. By definition,

$$\begin{aligned} & g_{\mu\nu}U^\mu = T_{\alpha\nu}Z^\alpha \\ i.e.\quad & U^\mu = g^{\mu\nu}T_{\alpha\nu}Z^\alpha \\ i.e.\quad & U^\mu = T^\mu_\alpha Z^\alpha\,. \end{aligned}$$

Thus, by the dominant energy condition, U is causal past directed. Moreover, by the Einstein equation,

$$T = \frac{1}{k}(\mathrm{Ric} - \frac{1}{2}Rg + \lambda g)$$

thus, as Z is null, and by definition of U, and by (4.2),

$$g(U, Z) = T(Z, Z) = \frac{1}{k}(\mathrm{Ric}(Z, Z) - \frac{1}{2}Rg(Z, Z) + \lambda g(Z, Z)) = 0\,.$$

This shows that $U \in Z^\perp \cap \mathsf{C}$ where C is the causal cone. By Lemma 4.2, U is proportional to Z, thus $T(Z, \cdot\,)$ is proportional to $g(Z, \cdot\,)$. We conclude, with the Einstein equation, that

$$\mathrm{Ric}(Z, \cdot\,) = kT(Z, \cdot\,) + \frac{1}{2}Rg(Z, \cdot\,) - \lambda g(Z, \cdot\,)$$

is proportional to $g(Z, \cdot\,)$, which is identically zero on $\mathcal{H}$ as Z is normal to $\mathcal{H}$. □

Proposition 4.3

Let (V, g) *be a Minkowski space and* $v \in V$ *a null vector. For any g-orthogonal spacelike vectors* $(b_1, \dots, b_{n-1})$ *of* $v^\perp$*, there is a null vector* $w \in V$ *such that* $(v, w, b_1, \dots, b_{n-1})$ *is a basis of* V*,* w *is normal to* $b_1, \dots, b_{n-1}$*, and* $g(v, w) = -1$.

Proof. Let $H = v^\perp$. Let P be a subspace of H supplementary to $\mathbb{R}v$ in H. The metric on P is non-degenerate, as P cannot contain any non-zero null vector, and has to be spacelike because otherwise P would contain independant a null vector, as shown in [ONe83, Lemma 5.27].

Let $b_1, \dots, b_{n-1}$ be an orthonormal basis of (P, g). As shown in [Min19, p. 168], as P is a codimension 2 subspace that does not intersect the null cone of V, and by convexity of the null cone, there are two different hyperplanes $H_\pm$ containing P and tangent to the null cone, thus containing null vectors $w_\pm$. By definition of P, one of those null vectors, say n_+, is v. Let $w := n_-$. Lemma 4.2 shows that $w^\perp$ is the hyperplane of V tangent to the null cone on $\mathbb{R}w$, thus $w^\perp = H_-$. v and w cannot be linearly dependent otherwise $H_- = H_+$, which shows that $(v, w, b_1, \dots, b_{n-1})$ is a basis of V, with v, w orthogonal to $b_1, \dots, b_{n-1}$. Moreover, by the demonstration of Proposition 2.1, as v and w are nulls, they cannot be orthogonal otherwise they would be linearly dependent, thus we can renormalise w to get $g(v, w) = -1$. □

We can now prove Theorem 4.1.

Proof of Theorem 4.1. Let $Y, W \in \Gamma\mathcal{H}$. By definition of ω_Z, the following equalities hold :

$$\begin{aligned}
\nabla_W \nabla_Y Z &= \nabla_W(\omega_Z(Y)Z) = \omega_Z(W)\omega_Z(Y)Z + W(\omega_Z(Y))Z \\
\nabla_Y \nabla_W Z &= \nabla_Y(\omega_Z(W)Z) = \omega_Z(W)\omega_Z(Y)Z + Y(\omega_Z(W))Z \\
\nabla_{[Y,W]} Z &= \omega_Z([Y,W])Z \; .
\end{aligned}$$

Thus, by definition of the curvature tensor R,

$$\begin{aligned}
R(W,Y)Z &= \nabla_W \nabla_Y Z - \nabla_Y \nabla_W Z - \nabla_{[Y,W]} Z \\
&= (W(\omega_Z(Y)) - Y(\omega_Z(W)) - \omega_Z([Y,W]))Z \; .
\end{aligned}$$

By the classic formula for the exterior derivative of a one-form, we get

$$R(W,Y)Z = \mathrm{d}\omega_Z(W,Y)Z \; . \tag{4.3}$$

Note that this equation only holds on $\mathcal{H}$ for vector fields Y, W tangent to $\mathcal{H}$. Let $e_1, \ldots, e_{n-1}$ be vectors tangent to $\mathcal{H}$ such that $(e_0 = Z, e_1, \ldots, e_{n-1})$ is a basis of $T_p\mathcal{H}$. In this basis, the coordinate of $\mathrm{d}\omega_Z(e_i, Y)Z$ on e_i is zero, except for $i = 0$, where it is $\mathrm{d}\omega_Z(Z, Y)$. Thus,

$$\mathrm{Tr} \begin{pmatrix} T_p\mathcal{H} & \longrightarrow & T_p\mathcal{H} \\ W & \longmapsto & \mathrm{d}\omega_Z(W,Y)Z \end{pmatrix} = \mathrm{d}\omega_Z(Z,Y)$$

which shows that

$$\mathrm{d}\omega_Z(Z,Y) = \mathrm{Tr} \begin{pmatrix} T_p\mathcal{H} & \longrightarrow & T_p\mathcal{H} \\ W & \longmapsto & R(W,Y)Z \end{pmatrix} \; . \tag{4.4}$$

Recall that by definition,

$$\mathrm{Ric}(Z,Y) = \mathrm{Tr} \begin{pmatrix} T_p\mathcal{M} & \longrightarrow & T_p\mathcal{M} \\ W & \longmapsto & R(W,Y)Z \end{pmatrix} \; .$$

We will now show that the traces on $\mathcal{H}$ and on $\mathcal{M}$ are equal. As in Proposition 4.3, let $b_1, \ldots, b_{n-1} \in T_p\mathcal{H}$, $N \in T_p\mathcal{M}$ null, such that $(Z, N, b_1, \ldots, b_{n-1})$ is a basis of $T_p\mathcal{M}$, such that N is orthogonal to $b_1, \ldots, b_{n-1}$, such that the b_i's are orthonormals, and such that $g(Z, N) = -1$. By definition of the trace, we only need to show that the coordinate of $R(N, Y)Z$ on N in this basis is 0.
Let $\nu, \mu, \eta^1, \ldots, \eta^{n-1} \in \mathbb{R}$ such that

$$R(N,Y)Z = \nu N + \mu Z + \sum_{i=1}^{n-1} \eta^i b_i \; .$$

We can now compute :

$$-g(Z, R(N,Y)Z) = -\nu g(Z,N) - \mu g(Z,Z) - \sum_{i=1}^{n-1} \eta^i g(Z, b_i) = \nu \; .$$

(which shows that the in the dual basis of $(Z, N, b_1, \ldots, b_{n-1})$, the dual of N is $-g(Z, \cdot)$.) Moreover, in local coordinates $(x^\mu)_{0 \le \mu \le n}$, we can compute

$$\begin{aligned}
g(Z, R(N,Y)Z) &= g_{\mu\nu} Z^\mu W^\alpha N^\beta Z^\gamma R^\nu_{\ \gamma\alpha\beta} \\
&= W^\alpha N^\beta Z^\gamma Z^\mu R_{\mu\gamma\alpha\beta} \; .
\end{aligned}$$

But, by swapping the indices γ and μ, and by antisymmetry of the curvature tensor in the first two indices $R_{\gamma\mu\alpha\beta} = -R_{\mu\gamma\alpha\beta}$ (see for example [HE73, p. 41]) , we get

$$\begin{aligned} Z^\gamma Z^\mu R_{\mu\gamma\alpha\beta} &= Z^\gamma Z^\mu R_{\gamma\mu\alpha\beta} \\ &= -Z^\gamma Z^\mu R_{\mu\gamma\alpha\beta} \end{aligned}$$

which shows that $Z^\gamma Z^\mu R_{\mu\alpha\beta\gamma}$, hence μ, is zero. Thus, by (4.4),

$$\mathrm{d}\omega_Z(Z,Y) = \mathrm{Ric}(Z,Y)\ .$$

We now conclude by Proposition 4.2 that $\mathrm{d}\omega_Z(Z,Y)$ is zero. □

5 Symmetry of non-degenerate compact Cauchy horizons

As introduced in Section 3, we will now show that the surface gravity of a non-degenerate compact Cauchy horizon can be normalised to -1. Before showing the future-incompleteness and the smoothness of the affine length, we present the *ribbon argument*, that was first introduced in [MI08], but which is modified here using a *horizontal lift*. We fix in this section $\mathcal{H}$ a non-degenerate compact Cauchy horizon in a spacetime satisfying the dominant energy condition.

5.1 The ribbon argument

Let Z be a future-directed null nowhere-zero vector field on $\mathcal{H}$ tangent to $\mathcal{H}$. Denote $\varphi : \mathcal{H}\times\mathbb{R} \longrightarrow \mathcal{H}$ the flow of Z and for $p \in \mathcal{H}$, the curve $\varphi_p := \varphi(p,\cdot\,) : \mathbb{R} \longrightarrow \mathcal{H}$. Also denote Z^* the smooth one form on $\mathcal{H}$ such that $Z^*(Z) = 1$ and $Z^*|_H = 0$, where H is a horizontal distribution. For ω a one-form on $\mathcal{H}$, $p \in \mathcal{H}$ and $\rho \geq 0$, we denote when possible

$$\int_{\varphi_p} \omega = \int_0^\infty \omega(Z(\varphi(p,z)))\,\mathrm{d}z$$

and

$$\int_{\varphi_p}^{\rho} \omega := \int_{\varphi_p|_{[0,\rho]}} \omega = \int_0^\rho \omega(Z(\varphi(p,z)))\,\mathrm{d}z$$

the integral of ω on φ_p up to ρ. We also define

$$\mathscr{F} := \left\{(p,e_1,\dots,e_n)\ \middle|\ p\in\mathcal{M},\ e_i\in H_p,\ (e_1,\dots,e_n) \text{ is an orthonormal basis of } H_p\right\}$$

the frame bundle of H.

Proposition 5.1

Let $(p_0,e_1,\dots,e_{n-1}) \in \mathscr{F}$. There is $\varepsilon > 0$ and $\eta > 0$ such that the map

$$\begin{array}{rccl} \psi^{p,(e_i)}: & \mathbb{R}^n & \longrightarrow & \mathcal{H} \\ & (x^1,\dots,x^{n-1},z) & \longmapsto & \varphi(\overline{\exp}(p_0,x^ie_i),z) \end{array}$$

restricted to $B(0,\varepsilon)\times(-\eta,\eta)$ is a diffeomorphism onto its image.

Notice that the image $\psi^{p,(e_i)}(B(0,\varepsilon)\times(-\eta,\eta))$ doesn't depend on the chosen orthonormal basis (e_i) of H_p.

Proof. By the inverse function theorem, we only need to show that the differential of $\psi^{p,(e_i)}$ in $0\in\mathbb{R}^n$ is an isomorphism. We can compute, for $1\leq i\leq n-1$,

$$\begin{aligned} \mathrm{d}\psi_0^{p,(e_i)}(\partial_z) &= \frac{\mathrm{d}}{\mathrm{d}z}\left[\varphi(\overline{\exp}(p_0,0),z)\right]_{z=0} = \frac{\mathrm{d}}{\mathrm{d}z}\left[\varphi(p_0,z)\right]_{z=0} = Z(p_0) \\ \mathrm{d}\psi_0^{p,(e_i)}(\partial_i) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\varphi(\overline{\exp}(p_0,te_i),0)\right]_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\overline{\exp}(p_0,te_i)\right]_{t=0} = e_i \end{aligned}$$

by Proposition 2.6. Thus, the image by $d\psi_0^{p,(e_i)}$ of the basis $(\partial_1, ..., , \partial_{n-1}, \partial_z)$ of $T_0\mathbb{R}^n$ is the basis $(e_1, ..., e_{n-1}, Z(p_0))$ of $T_{p_0}\mathcal{H}$, hence the result. □

Proposition 5.2

Let $\alpha : [0,1] \longrightarrow \mathcal{H}$ be a horizontal curve with nowhere-zero velocity, and denote $\alpha(0) = p$. For every $\rho \geq 0$, there is a unique horizontal curve with nowhere-zero velocity $\alpha_\rho : [0,1] \longrightarrow \mathcal{H}$ such that

(i) $\alpha_\rho(0) = \varphi(p, \rho)$

(ii) $\alpha_\rho(t) \in \varphi_{\alpha(t)}$.

The curve α_ρ is called the ρ-horizontal lift of α.

Proof. If it exists, α_ρ has to be of the form $\alpha_\rho(t) = \varphi(\alpha(t), f(t))$ with $f(0) = \rho$. We can compute

$$\alpha'_\rho(t) = \mathrm{d}(\varphi_{f(t)})_{\alpha(t)}(\alpha'(t)) + f'(t)Z(\varphi(\alpha(t), f(t))) .$$

Thus, as α_ρ is horizontal, f satisfies the ODE

$$\begin{cases} f'(t) = -Z^*(\mathrm{d}(\varphi_{f(t)})_{\alpha(t)}(\alpha'(t))) \\ f(0) = \rho \end{cases}$$

which has a unique solution, by ODE theory. Hence the uniqueness. For the existence, defining α_ρ by the formula above where f is the solution of the previous ODE gives a horizontal curve projecting on α with the right initial condition, the only thing left to prove is that it has a nowhere zero velocity. Suppose that $\alpha'_\rho(t) = 0$, *i.e.* that $\mathrm{d}(\varphi_{f(t)})_{\alpha(t)}(\alpha'(t)) \in \mathbb{R}Z$. Then $\alpha'(t) = \mathrm{d}\varphi_{-f(t)}(\mathrm{d}(\varphi_{f(t)})_{\alpha(t)}(\alpha'(t))) \in \mathbb{R}Z \cap H$, which implies that $\alpha'(t) = 0$ which is absurd. □

Lemma 5.1

For every null nowhere-zero vector field Z,

$$\mathcal{L}_Z g|_{T\mathcal{H}} = 0 .$$

Proof. As $\mathcal{L}_Z$ is a tensor derivation, for $X, Y \in \Gamma\mathcal{H}$,

$$\begin{aligned} \mathcal{L}_Z g(X,Y) &= \mathcal{L}_Z(g(X,Y)) - g(\mathcal{L}_Z X, Y) - g(X, \mathcal{L}_Z Y) \\ &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y) - g([Z,X], Y) - g(X, [Z,Y]) \\ &= g(\nabla_X Z, Y) + g(X, \nabla_Y Z) \\ &= 0 \end{aligned}$$

because $\nabla_X Z = \omega_Z(X)Z$ and $\nabla_Y Z = \omega_Z(Y)Z$ are null vector fields. □

Proposition 5.3

Let $p, p' \in \mathcal{H}$ such that there is $z, z' \in (-\eta, \eta)$ and X in the open ball centered in 0 of radius $\varepsilon > 0$ in $H_{\varphi(p,z)}$ such that $\varphi(p', z') = \overline{\exp}(\varphi(p,z), X)$. Then there is a smooth function ϕ such that $\phi(p, p', \cdot) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is strictly increasing, and such that for every $\rho \geq 0$,

$$\left| \int_{\varphi_p}^{\rho} \omega_Z - \int_{\varphi_{p'}}^{\phi(p,p',\rho)} \omega_Z \right| \leq 2K\varepsilon + 2\eta \|\omega_Z(Z)\|_{L^\infty(\mathcal{H})}$$

where K is a global constant independent of p and p'.

Proof. Denote $p_0 = \varphi(p, z)$ and $p_1 = \varphi(p', z')$. By definition, the horizontal geodesic

$$\begin{array}{rccc} \alpha : & [0,1] & \longrightarrow & \mathcal{U} \\ & t & \longmapsto & \overline{\exp}(p_0, tX) \end{array}$$

satisfies $\alpha(0) = p_0$ and $\alpha(1) = p_1$. Denote $\alpha_\rho : [0,1] \longrightarrow \mathcal{H}$ the ρ-horizontal lift of α, given by Proposition 5.2. More precisely, as seen in the proof of Proposition 5.2, we have $\alpha_\rho(t) = \varphi(\alpha(t), f_\rho(t))$ where $f_\rho : [0,1] \longrightarrow \mathbb{R}$ is the unique solution of the ODE

$$\begin{cases} (f_\rho)'(t) = -Z^*(\mathrm{d}(\varphi_{f_\rho(t)})_{\alpha_\rho(t)}((\alpha_\rho)'(t))) \\ f_\rho(0) = \rho \ . \end{cases}$$

Let ϕ be the function defined on a neighbourhood of (p, p') times $\mathbb{R}_+$ by

$$\phi(p, p', \rho) = f_\rho(1) \ .$$

Notice that by ODE theory, ϕ is smooth. Moreover, by construction,

$$\varphi(p_1, \phi(p, p', \rho)) = \alpha_\rho(1) \ .$$

Now, define the map

$$\begin{array}{rccc} R : & [0,1] \times \mathbb{R}_+ & \longrightarrow & \mathcal{H} \\ & (s, \rho) & \longmapsto & \alpha_\rho(s) \end{array} .$$

We can compute

$$\frac{\partial R}{\partial \rho} = \frac{\partial f_\rho(1)}{\partial \rho} Z(\alpha_\rho(s)) \in \mathbb{R}Z \backslash \{0\}$$

$$\frac{\partial R}{\partial s} = (\alpha_\rho)'(s) \in H \backslash \{0\} \ .$$

Notice that $\partial f_\rho(0)/\partial\rho = 1$, thus reducing $\varepsilon > 0$ if necessary, $\partial f_\rho(1)/\partial \rho > 0$, in which case R is a smooth immersion, called the *Ribbon*, and in which case $\phi(p, p', \cdot)$ is strictly increasing. Denote $R_\rho = R([0,1] \times [0, \rho])$. By definition, R_ρ is an immersed corner submanifold of $\mathcal{H}$, with boundary the union of the null curves $\varphi_{p_0}([0, \rho])$, $\varphi_{p_1}([0, \phi(p_0, p, \rho)])$, and of the horizontal curves α and α_ρ.

Lemma 5.2

$$\int_{R_\rho} \mathrm{d}\omega_Z = 0 \ .$$

Proof. In fact, $\mathrm{d}\omega_Z$ is zero on the tangent bundle of R_ρ. Indeed, R_ρ is 2-dimensional and tangent to the null direction. Let $x \in R_\rho$ and split $T_x R_\rho = \mathbb{R}Z(x) + \mathbb{R}Y$. Then for $X_1 = a_1 Z(x) + b_1 Y$, $X_2 = a_2 Z(x) + b_2 Y \in T_p R_\rho$, we have by antisymmetry

$$\mathrm{d}\omega(X_1, X_2) = (a_1 b_2 - b_1 a_2)\mathrm{d}\omega(Z(x), Y) = 0$$

because ω is null-closed by Section 4. ☐

As proved in [Lee83, Th 16.25], Stokes' theorem holds for corner manifolds. Thus,

$$\int_{\varphi_{p_0}}^{\rho} \omega_Z = \int_{\varphi_{p_1}}^{\phi(p,p',\rho)} \omega_Z + \int_0^1 \omega_Z((\alpha_\rho)'(s))\mathrm{d}s - \int_0^1 \omega_Z((\alpha)'(s))\mathrm{d}s \ . \tag{5.1}$$

The following result is a fundamental property of horizontal distributions.

Lemma 5.3

There is a global constant $K > 0$ such that for every horizontal curve $\alpha : [0,1] \longrightarrow \mathcal{H}$,

$$\left| \int_0^1 \omega_Z(\alpha'(s))\mathrm{d}s \right| \leq K \sup_{s\in[0,1]} \sqrt{g(\alpha'(s),\alpha'(s))} \,.$$

Proof. As shown in Proposition A.3, the unitary subbundle $\mathscr{U}$ of the horizontal distribution H is compact, on which ω_Z is a continuous function. Thus, there is a constant $K > 0$ such that for $X \in \mathscr{U}$, $\omega_Z(X) \leq K$.
Then, if $X \in H$ is any horizontal vector, $\omega_Z(X) \leq K\sqrt{g(X,X)}$. This shows that

$$\begin{aligned} \left| \int_0^1 \omega_Z(\alpha'(s))\mathrm{d}s \right| &\leq \int_0^1 |\omega_Z(\alpha'(s))|\,\mathrm{d}s \\ &\leq K \int_0^1 \sqrt{g(\alpha'(s),\alpha'(s))}\mathrm{d}s \\ &\leq K \sup_{s\in[0,1]} \sqrt{g(\alpha'(s),\alpha'(s))} \,. \end{aligned}$$

□

Notice that for $\rho \geq 0$, by definition of α_ρ,

$$\alpha'_\rho(s) = \mathrm{d}(\varphi_{f_\rho(t)})(\alpha'(t)) + (f_\rho)'(t)Z(\varphi(\alpha(t), f_\rho(t))) = \pi(\mathrm{d}(\varphi_{f_\rho(t)})(\alpha'(t)))$$

where π is the projection $T\mathcal{H} = H \oplus \mathbb{R}Z \longrightarrow H$. This shows that

$$g(\alpha'_\rho(s), \alpha'_\rho(s)) = g(\mathrm{d}(\varphi_{f_\rho(t)})(\alpha'(t)), \mathrm{d}(\varphi_{f_\rho(t)})(\alpha'(t))) \,. \tag{5.2}$$

Lemma 5.1 shows that for any $v \in T\mathcal{H}$, the function

$$r \longmapsto g(\mathrm{d}(\varphi_r)(v), \mathrm{d}(\varphi_r)(v))$$

is constant. Indeed, as $\mathcal{L}_Z g = 0$ on $\mathcal{H}$ by Lemma 5.1,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[g(\mathrm{d}(\varphi_r)(v), \mathrm{d}(\varphi_r)(v))\right] = 2g(\mathcal{L}_Z \mathrm{d}(\varphi_r)(v), \mathrm{d}(\varphi_r)(v)) = 0 \,.$$

Applying this result with $v = \alpha'(t)$ and using (5.2) shows that for every $\rho \geq 0$ and $s \in [0,1]$,

$$g(\alpha'_\rho(s), \alpha'_\rho(s)) = g(\alpha'(s), \alpha'(s)) = g(\alpha'(0), \alpha'(0))$$

where the second equality holds because α is a horizontal geodesic, thus

$$\frac{\mathrm{d}}{\mathrm{d}s}[g(\alpha'(s), \alpha'(s))] = 2g(\nabla'_\alpha \alpha'(s), \alpha'(s)) = 2g(\pi(\nabla'_\alpha \alpha'(s)), \alpha'(s)) = 0 \,.$$

Recall that by definition of the horizontal exponential map, $\alpha'(0) = X$ where X is in the ball of radius ε. Thus, $g(\alpha'_\rho(s), \alpha'_\rho(s)) \leq \varepsilon^2$.
This fact, Equation (5.1) and Lemma 5.3 combined show that

$$\left| \int_{\varphi_{p_0}}^{\rho} \omega_Z - \int_{\varphi_{p_1}}^{\phi(p,p',\rho)} \omega_Z \right| \leq 2K\varepsilon \,.$$

We can now conclude :

$$\begin{aligned} \left| \int_{\varphi_p}^{\rho} \omega_Z - \int_{\varphi_{p'}}^{\phi(p,p',\rho)} \omega_Z \right| &\leq \left| \int_{\varphi_{p_0}}^{\rho} \omega_Z - \int_{\varphi_{p_1}}^{\phi(p,p',\rho)} \omega_Z \right| + \left| \int_{\varphi_p}^{z} \omega_Z \right| + \left| \int_{\varphi_{p'}}^{z'} \omega_Z \right| \\ &\leq 2K\varepsilon + 2\eta \|\omega_Z(Z)\|_{L^\infty(\mathcal{H})} \,. \end{aligned}$$

□

Denote $\phi : \mathcal{X} \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ the function introduced in the previous proof. It is defined on $\mathcal{X} \times \mathbb{R}_+$, where $\mathcal{X} \subseteq \mathcal{H}^2$ is the set of pairs (p, p') as in Proposition 5.3. We saw in the previous proof that ϕ is smooth, as $\phi(p, p', \rho)$ is the end point of an ODE with parameters depending smoothly on p, p', ρ.

Proposition 5.4

For every $p_0 \in \mathcal{H}$, there is a constant $C > 0$, an open neighbourhood $\mathcal{U} \subseteq \mathcal{H}$ of p_0, such that for every $p, p' \in \mathcal{U}$,

$$\left| \int_{\varphi_p}^{\rho} \omega_Z - \int_{\varphi_{p'}}^{\phi(p,p',\rho)} \omega_Z \right| \leq C$$

This result, which contains the essence of the ribbon argument, will be used in the following section to show future incompleteness of the future generators of $\mathcal{H}$.

Proof. Let $\varepsilon > 0$ and $\eta > 0$ be as in Proposition 5.1 and denote $\mathcal{U}$ the associated cylinder open neighbourhood of p_0. Let $C = 2K\varepsilon + 2\eta \|\omega_Z(Z)\|_{L^\infty(\mathcal{H})}$. Then by definition of the map $\psi^{p_0,(e_i)}$, for every $p, p' \in \mathcal{U}$, the pair (p, p') satisfies the hypothesis of Proposition 5.3. Indeed, there is $X = x^i e_i$ where $(x^i) \in B(0, \varepsilon)$ and $z, z' \in (-\eta, \eta)$ such that $p' = \varphi(\overline{\exp}(\varphi(p, z), X), z')$. Hence the result, applying Proposition 5.3. □

5.2 Future-incompleteness of the null generators

Denote $\mathcal{I}$ the set of points $p \in \mathcal{H}$ such that the null generators leaving p are future-incomplete. As $\mathcal{H}$ is non-degenerate, $\mathcal{I} \neq \varnothing$. Let Z be a future-directed null nowhere-zero vector field on $\mathcal{H}$ tangent to $\mathcal{H}$, and denote $L_Z : \mathcal{H} \longrightarrow \mathbb{R} \cup \{\infty\}$ the affine length associated to Z. Then $\mathcal{I}$ is the set of points $p \in \mathcal{H}$ such that $L_Z(p) < \infty$. As before, we denote $\varphi : \mathcal{H} \times \mathbb{R} \longrightarrow \mathcal{H}$ the flow of Z and for $p \in \mathcal{H}$, the curve $\varphi_p := \varphi(p, \cdot\,) : \mathbb{R} \longrightarrow \mathcal{H}$.

Proposition 5.5

Let X be a future-directed smooth null nowhere-zero vector field on $\mathcal{H}$ tangent to $\mathcal{H}$, and denote φ^X its flow. For $p \in \mathcal{H}$, the affine length $L_X(p)$ of the null generator γ leaving p in $t = 0$ with velocity $\gamma'(0) = X(p)$ is given by the formula

$$L_X(p) = \int_0^\infty \exp\left(\int_{\varphi_p^X}^{\rho} \omega_X \right) \mathrm{d}\rho\ .$$

Proof. Denote $\psi : [0, +\infty) \longrightarrow [0, L_X(p))$ the reparametrization of φ_p^X as a geodesic, *i.e.* such that $\gamma \circ \psi = \varphi_p^X$. We have $\psi'(t)\gamma'(\psi(t)) = X(\varphi_p^X(t))$. Denote $\phi = 1/\psi'$. The geodesic equation reads

$$0 = \nabla_{\gamma'}\gamma' = \phi(\phi' + \phi\, \omega_X(X))X$$

thus

$$\phi(\rho) = \exp\left(-\int_0^\rho \omega_X(X(\varphi_p^X(s)))\mathrm{d}s \right) = \exp\left(-\int_{\varphi_p^X}^{\rho} \omega_X \right)\ .$$

This shows

$$\psi(t) = \int_0^t \exp\left(\int_{\varphi_p^X}^{\rho} \omega_X \right) \mathrm{d}\rho$$

hence the result, taking the limit $t \to +\infty$. □

The next result shows how the surface gravity changes when using a different Z.

Proposition 5.6

For every smooth strictly positive function $f : \mathcal{H} \longrightarrow \mathbb{R}_+^*$,

$$\omega_{fZ} = \omega_Z + \mathrm{d}(\log f) \quad \textit{and} \quad \omega_{fZ}(fZ) = f\omega_Z(Z) + \partial_Z f \ .$$

Proof. As $\mathrm{d}(\log f) = \mathrm{d}f/f$, the second equation follows directly from the first one. By definition,

$$\begin{aligned} \omega_{fZ}(X) fZ &= \nabla_X fZ \\ &= (\partial_X f)Z + f\nabla_X Z \\ &= \left(\frac{\partial_X f}{f} + \omega_Z(X)\right) fZ \\ &= (\mathrm{d}(\log f)(X) + \omega_Z(X)) fZ \end{aligned}$$

hence the result. □

Lemma 5.4

For every $p_0 \in \mathcal{I}$, $\displaystyle\int_{\varphi_{p_0}} \omega_Z = -\infty$.

Proof. As $L_Z(p_0) < \infty$, the function

$$\rho \longmapsto \exp\left(\int_{\varphi_{p_0}}^{\rho} \omega_Z\right) = \exp\left(\int_0^{\rho} \omega_Z(Z(\varphi(p_0, z)))\mathrm{d}z\right)$$

is integrable on $\mathbb{R}_+$. A classical result that can be found for example in [Les10] shows that for almost every $x > 0$,

$$\exp\left(\int_0^{nx} \omega_Z(Z(\varphi(p_0, z)))\mathrm{d}z\right) \xrightarrow[n\to\infty]{} 0 \ .$$

Thus, there is $x > 0$ such that

$$\int_0^{nx} \omega_Z(Z(\varphi(p_0, z)))\mathrm{d}z \xrightarrow[n\to\infty]{} -\infty \ .$$

Now, if $\rho \geq 0$, write the unique decomposition

$$\rho = [\rho]_x + \eta(\rho)$$

with $[\rho]_x \in \mathbb{N}x$ and $\eta(\rho) \in [0, x)$ (if $x = 1$, this is the classical decomposition with the integer and fractional parts of ρ). We can compute

$$\int_0^{\rho} \omega_Z(Z(\varphi(p_0, z)))\mathrm{d}z = \int_0^{[\rho]_x} \omega_Z(Z(\varphi(p_0, z)))\mathrm{d}z + \int_{[\rho]_x}^{[\rho]_x+\eta(\rho)} \omega_Z(Z(\varphi(p_0, z)))\mathrm{d}z \xrightarrow[\rho\to\infty]{} -\infty$$

as by choice of x,

$$\int_0^{[\rho]_x} \omega_Z(Z(\varphi(p_0, z)))\mathrm{d}z \xrightarrow[\rho\to\infty]{} -\infty$$

and as

$$\left|\int_{[\rho]_x}^{[\rho]_x+\eta(\rho)} \omega_Z(Z(\varphi(p_0, z)))\mathrm{d}z\right| \leq x\|\omega_Z(Z)\|_{L^\infty(\mathcal{H})}$$

is bounded, because $\omega_Z(Z)$ is bounded on $\mathcal{H}$ by compactness. □

We use the ribbon argument to for the next proposition.

Proposition 5.7

For every $p \in \mathcal{H}$, $\displaystyle\int_{\varphi_p} \omega_Z = -\infty$.

Proof. Define $\mathcal{A}$ as the set of points $p \in \mathcal{H}$ such that $\int_{\varphi_p} \omega_Z = -\infty$. As $\mathcal{H}$ is non-degenerate, by Lemma 5.4, we know that $\mathcal{A} \neq \varnothing$. Let us use the Ribbon argument to show that $\mathcal{A}$ is open and closed in $\mathcal{H}$.
Let $p_0 \in \mathcal{A}$. Denote ϕ the function defined in Section 5.1 and $\mathcal{U}$ the open set given by Proposition 5.4. Let $p \in \mathcal{U}$. We can write

$$\int_{\varphi_{p_0}}^{\rho} \omega_Z = F(p_0, p, \rho) + \int_{\varphi_p}^{\phi(p_0,p,\rho)} \omega_Z \tag{5.3}$$

where F is a bounded function. If $\rho \longmapsto \phi(p_0, p, \rho)$ didn't diverge to $+\infty$, there would be a bounded sequence $\phi(p_0, p, \rho_n)$ with $\rho_n \longrightarrow +\infty$, but then the sequence $\int_{\varphi_{p_0}}^{\rho_n} \omega_Z$ would be bounded which would is absurd as $p_0 \in \mathcal{A}$. Thus, $\rho \longmapsto \phi(p_0, p', \rho)$ diverges to $+\infty$. As F is bounded, taking the limit $\rho \to \infty$ of (5.3) shows that $\int_{\varphi_p} \omega_Z = -\infty$, *i.e.* $p \in \mathcal{A}$. This shows that $\mathcal{A}$ is open in $\mathcal{H}$.
Let $p \in \overline{\mathcal{A}}$ and let $(p_n)_{n\geq 1} \subset \mathcal{A}$ be a sequence converging to p. Let $\mathcal{U}$ the open neighbourhood of p given by Proposition 5.4. Let $N \in \mathbb{N}$ be large enough such that $p_0 := p_N \in \mathcal{U}$. Then we can write again

$$\int_{\varphi_{p_0}}^{\rho} \omega_Z = F(p_0, p, \rho) + \int_{\varphi_p}^{\phi(p_0,p,\rho)} \omega_Z \tag{5.4}$$

where F is a bounded function. The exact same reasoning as above applies : if $\rho \longmapsto \phi(p_0, p, \rho)$ didn't diverge to $+\infty$, there would be a bounded sequence $\phi(p_0, p, \rho_n)$ with $\rho_n \longrightarrow +\infty$, but then the sequence $\int_{\varphi_{p_0}}^{\rho_n} \omega_Z$ would be bounded which would is absurd as $p_0 \in \mathcal{A}$. Thus, $\rho \longmapsto \phi(p_0, p, \rho)$ diverges to $+\infty$. As F is bounded, taking the limit $\rho \to \infty$ of (5.3) shows that $\int_{\varphi_p} \omega_Z = -\infty$, *i.e.* $p \in \mathcal{A}$.
Thus, $\mathcal{A} \neq \varnothing$ is open and closed in $\mathcal{H}$ which is connected *i.e.* $\mathcal{A} = \mathcal{H}$, hence the result. ☐

Lemma 5.5

Every $p_0 \in \mathcal{H}$ admits an open neighbourhood $\mathcal{U}$ and a constant $C > 0$ such that for every $\rho \geq C$, and for every $p \in \mathcal{U}$,

$$\int_{\varphi_p}^{\rho} \omega_Z < 0 \, .$$

Proof. Let $\mathcal{U}$ the open neighbourhood of p_0 given by Proposition 5.4. We can write

$$\int_{\varphi_p}^{\rho} \omega_Z = F(p, p_0, \rho) + \int_{\varphi_{p_0}}^{\phi(p,p_0,\rho)} \omega_Z \, .$$

Where F is bounded. Let $R > 0$ such that $|F| \leq R$. We know that $\rho \longmapsto \phi(p, p_0, \rho)$ diverges to $+\infty$, otherwise there would be a bounded sequence $\int_{\varphi_p}^{\rho_n} \omega_Z$ with $\rho_n \to +\infty$ which is absurd by Proposition 5.7. Moreover, for $p \in K$,

$$\int_{\varphi_p}^{\rho} \omega_Z \leq \int_{\varphi_{p_0}}^{\phi(p,p_0,\rho)} \omega_Z + R \, . \tag{5.5}$$

By Proposition 5.7, there is a constant $C_0 > 0$ such that for every $\rho \geq C_0$,

$$\int_{\varphi_{p_0}}^{\rho} \omega_Z < -R \, .$$

Let $C := C_0 + 1$. As $\phi(p_0, p_0, C) = C > C_0$, by continuity, for p in an open neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of p_0, $\phi(p, p_0, C) > C_0$. Then, as $\rho \longmapsto \phi(p, p_0, \rho)$ is increasing, by (5.5), for every $\rho > C$ and every $p \in \mathcal{V}$,

$$\int_{\varphi_p}^{\rho} \omega_Z \leq \int_{\varphi_{p_0}}^{\phi(p,p_0,\rho)} \omega_Z + R < -R + R = 0 \ .$$

Hence the result, with the neighbourhood $\mathcal{V}$ of p_0 and the constant $C > 0$. □

The next proposition is a direct consequence of Lemma 5.5 and of the compactness of $\mathcal{H}$.

Proposition 5.8

There is a constant $C > 0$ such that for every $p \in \mathcal{H}$ and $\rho \geq C$,

$$\int_{\varphi_p}^{\rho} \omega_Z < 0 \ .$$

Proof. For $p \in \mathcal{H}$, denote $\mathcal{U}_p$ and C_p the open neighbourhood of p and the constant given by Lemma 5.5. By compactness, let $(\mathcal{U}_{p_i})_{1 \leq i \leq m}$ be a subcover of $\mathcal{H}$, and define $C = \max_{1 \leq i \leq m} C_{p_i} > 0$. Then if $p \in \mathcal{U}_{p_i}$, as $C > C_{p_i}$, if $\rho \geq C$, $\int_{\varphi_p}^{\rho} \omega_Z < 0$. Hence the result, as $\mathcal{H} = \bigcup_{1 \leq i \leq m} \mathcal{U}_{p_i}$. □

Proposition 5.9

There is a future-directed null nowhere-zero vector field X on $\mathcal{H}$ such that $\omega_X(X) < 0$.

Proof. Define the smooth strictly positive function $g : \mathcal{H} \longrightarrow \mathbb{R}_+^*$ by

$$g(p) = \int_0^C \exp\left(\int_{\varphi_p}^{\rho} \omega_Z\right) \mathrm{d}\rho$$

where $C > 0$ is the constant given by Proposition 5.8. We will show that the vector field $X := gZ$ satisfies $\omega_X(X) < 0$ by a simple computation. By Proposition 5.6,

$$\omega_X(X) = g\omega_Z(Z) + \partial_Z g \ .$$

Moreover, a classical theorem for the differentiation under the integral sign on a compact interval of a smooth function shows that

$$\begin{aligned}
\partial_Z g(p) &= \int_0^C \partial_Z \exp\left(\int_{\varphi_p}^{\rho} \omega_Z\right) \mathrm{d}\rho \\
&= \int_0^C \partial_Z \left[\int_{\varphi_p}^{\rho} \omega_Z\right] \exp\left(\int_{\varphi_p}^{\rho} \omega_Z\right) \mathrm{d}\rho \\
&= \int_0^C \left[\int_0^{\rho} \partial_Z[\omega_Z(Z(\varphi(p,z)))] \, \mathrm{d}z\right] \exp\left(\int_{\varphi_p}^{\rho} \omega_Z\right) \mathrm{d}\rho \ .
\end{aligned}$$

Moreover, by definition,

$$\partial_Z[\omega_Z(Z(\varphi(p,z)))] = \frac{\mathrm{d}}{\mathrm{d}t}\left[\omega_Z(Z(\varphi(\varphi(p,t),z)))\right]_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\omega_Z(Z(\varphi(p,t+z)))\right]_{t=0} = \frac{\mathrm{d}}{\mathrm{d}z}\left[\omega_Z(Z(\varphi(p,z)\right] \ .$$

Thus,

$$\begin{aligned}\partial_Z g(p) &= \int_0^C (\omega_Z(Z(\varphi(p,\rho))) - \omega_Z(Z(p))) \exp\left(\int_{\varphi_p}^{\rho} \omega_Z\right) \mathrm{d}\rho \\ &= \int_0^C \omega_Z(Z(\varphi(p,\rho))) \exp\left(\int_{\varphi_p}^{\rho} \omega_Z\right) \mathrm{d}\rho - \omega_Z(Z(p))g(p) \\ &= \int_0^C \frac{\mathrm{d}}{\mathrm{d}\rho}\left[\exp\left(\int_{\varphi_p}^{\rho} \omega_Z\right)\right] \mathrm{d}\rho - \omega_Z(Z(p))g(p) \\ &= \exp\left(\int_{\varphi_p}^{C} \omega_Z\right) - 1 - \omega_Z(Z(p))g(p) \ .\end{aligned}$$

This shows that for every $p \in \mathcal{H}$,

$$\omega_X(X)_p = \exp\left(\int_{\varphi_p}^{C} \omega_Z\right) - 1 < 0$$

as, by the choice of C, $\displaystyle\int_{\varphi_p}^{C} \omega_Z < 0$. ☐

Corollary 5.1

The null generators of $\mathcal{H}$ are future-incomplete.

Proof. Let X be the vector field given by Proposition 5.9. It is enough to show that the affine length $L_X(p)$ of the null geodesic leaving any $p \in \mathcal{H}$ with initial velocity $X(p)$ is finite. As $\omega_X(X) < 0$ is continous on the compact set $\mathcal{H}$, there is a constant $-K < 0$ such that $\omega_X(X) \leq -K$ (notice that $-K$ is simply the supremum for $p \in \mathcal{H}$ of the continuous function $\exp\left(\int_{\varphi_p}^{C} \omega_Z\right) - 1$).

We can now compute as usual, by Proposition 5.5,

$$L_X(p) = \int_0^\infty \exp\left(\int_0^\rho \omega_X(X(\varphi_X(p,z)))\mathrm{d}z\right) \mathrm{d}\rho \leq \int_0^\infty e^{-K\rho}\, \mathrm{d}\rho = \frac{1}{K} < \infty \ .$$

☐

5.3 The homogeneity vector field

Let $\mathcal{H}$ be a non-degenerate compact Cauchy horizon. In Section 5.2, we found a smooth null nowhere-zero vector field n on $\mathcal{H}$ such that, on $\mathcal{H}$, $\kappa := \omega(n) < -K$, where $K > 0$ is a constant. Denote $\omega := \omega_n$ and n^* the one-form on $\mathcal{H}$ such that $n^*(n) = 1$ and $n^*|_H = 0$. Corollary 5.1 shows that for every $p \in \mathcal{H}$, the affine length $\Lambda(p)$ of the null geodesic λ such that $\lambda(0) = p$, $\lambda'(0) = n(p)$ is finite. We saw that, denoting φ the flow of n, the function $\Lambda : \mathcal{H} \longrightarrow \mathbb{R}_+^*$ is defined by

$$\Lambda(p) = \int_0^\infty \exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right) \mathrm{d}\rho \ .$$

The goal of this section is to show that the affine length function is smooth. This was the second step of the proof strategy introduced in Section 3.3 to show the existence of a pre-Killing field on $\mathcal{H}$. The general proof that Λ is $\mathscr{C}^k$ goes by induction and is the object of Appendix C, but the first two steps are presented here as well, in order to clarify the general proof. As Λ is defined by an integral, to show its regularity we find locally uniform dominations of the integrated functions. We also show that Λ satisfies a PDE, that proves its smoothness in the vacuum case.

For $\rho \in \mathbb{R}$, denote $\varphi^\rho = \varphi(\,\cdot\,,\rho) : \mathcal{H} \longrightarrow \mathcal{H}$. In this section, N will denote a null vector field transverse to $\mathcal{H}$ such that $N \perp \ker\omega$ and such that $g(n, N) = -1$, as in Proposition 4.3. We extend N geodesically on a neighbourhood of $\mathcal{H}$, and we extend n on that same neighbourhood with the flow of N such that $[N, n] = 0$. We also extend κ such that $\partial_N \kappa = 0$.

Lemma 5.6

When restricted to $T\mathcal{H}$,

$$\mathcal{L}_n \omega = \mathrm{d}\kappa\ .$$

Proof. By Cartan's magic formula,

$$\mathcal{L}_n \omega = \mathrm{d}(\omega(n)) + \mathrm{d}\omega(n, \cdot\,) = \mathrm{d}\kappa$$

because, as seen in Section 4, ω is null-closed. □

The following result will be used to show the regularity of Λ.

Lemma 5.7

There is a constant $C > 0$ such that for every $(p, X) \in T\mathcal{H}$ and $\rho \geq 0$,

$$|\omega(\mathrm{d}\varphi^\rho(X))| \leq C\left(\sqrt{g(X,X)}(1 + \rho\|\kappa\|_{L^\infty(\mathcal{H})}) + |\lambda|\right)$$

where λ is the coordinate of X along n in the decomposition $T\mathcal{H} = \mathbb{R}n \oplus H$.

In other words, the function $\rho \longmapsto \omega(\mathrm{d}\varphi^\rho(X))$ is bounded by an affine function.

Proof. Let $(p, X) \in T\mathcal{H}$. For every $\rho \geq 0$, there is $X_H^\rho \in H$ and $\lambda(\rho)$, smooth in ρ, such that

$$\mathrm{d}\varphi^\rho(X) = X_H^\rho + \lambda(\rho) n(\varphi(p, \rho))\ . \tag{5.6}$$

Thus, we have

$$\omega(\mathrm{d}\varphi^\rho(X)) = \omega(X_H^\rho) + \lambda(\rho)\omega(n)_{\varphi(p,\rho)}\ .$$

By Proposition A.3, the unitary subbundle of H is compact. Thus, there is $C > 0$ such that for every $Y \in H$,

$$|\omega(Y)| \leq C\sqrt{g(Y,Y)}\ .$$

As $\mathcal{L}_n(\mathrm{d}\varphi^\rho(X)) = 0$ and $\mathcal{L}_n(\lambda(\rho)n(\varphi(p,\rho))) = \lambda'(\rho)n(\varphi(p,\rho))$, Equation (5.6) and the definition of n^* gives

$$\lambda'(\rho) = -n^*(\mathcal{L}_n X_H^\rho)\ .$$

Moreover, as $n^*(X_H^\rho) = 0$ on φ_p, we have

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}\rho}[n^*(X_H^\rho)] \\ &= \mathcal{L}_n[n^*(X_H^\rho)] \\ &= n^*(\mathcal{L}_n X_H^\rho) + \mathcal{L}_n n^*(X_H^\rho) \\ &= -\lambda'(\rho) + \mathcal{L}_n n^*(X_H^\rho) \end{aligned}$$

which shows that

$$\lambda'(\rho) = \mathcal{L}_n n^*(X_H^\rho)\ .$$

Define the smooth one-form $\beta = \mathcal{L}_n n^*$ on $\mathcal{H}$, such that $\lambda'(\rho) = \beta(X_H^\rho)$. Once again, as the unitary subbundle of H is compact, changing $C > 0$ if necessary, for every $Y \in H$, $|\beta(Y)| \leq C\sqrt{g(Y,Y)}$. Thus, as n is normal to $T\mathcal{H}$,

$$|\lambda'(\rho)| \leq C\sqrt{g(X_H^\rho, X_H^\rho)} = C\sqrt{g(\mathrm{d}\varphi^\rho(X), \mathrm{d}\varphi^\rho(X))}\ .$$

Moreover, as $\mathcal{L}_n[\mathrm{d}\varphi^\rho(X)] = 0$, and with Lemma 5.1, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\rho}[g(\mathrm{d}\varphi^\rho(X), \mathrm{d}\varphi^\rho(X))] &= \mathcal{L}_n(g(\mathrm{d}\varphi^\rho(X), \mathrm{d}\varphi^\rho(X)))\\ &= \mathcal{L}_n g(\mathrm{d}\varphi^\rho(X), \mathrm{d}\varphi^\rho(X)) + 2g(\mathcal{L}_n \mathrm{d}\varphi^\rho(X), \mathrm{d}\varphi^\rho(X))\\ &= 0\ .\end{aligned}$$

This shows that $g(\mathrm{d}\varphi^\rho(X), \mathrm{d}\varphi^\rho(X)) = g(X,X)$, thus $|\lambda'(\rho)| \leq C\sqrt{g(X,X)}$, and finally,

$$|\lambda(\rho)| \leq |\lambda(0)| + C\rho\sqrt{g(X,X)}\ .$$

To conclude, using the fact that changing $C > 0$ if necessary, $|\omega(n)_{\varphi(p,\rho)}| < C$, we get as wished

$$|\omega(\mathrm{d}\varphi^\rho(X))| \leq |\omega(X_H^\rho)| + |\lambda(\rho)||\omega(n)_{\varphi(p,\rho)}| \leq C\sqrt{g(X,X)}(1 + \rho|\omega(n)_{\varphi(p,\rho)}|) + |\lambda(0)\omega(n)_{\varphi(p,\rho)}|\ .$$

□

Proposition 5.10

The function Λ is $\mathscr{C}^1$ and satisfies, for $p \in \mathcal{H}$ and $X \in T_p\mathcal{H}$,

$$\mathrm{d}\Lambda_p(X) = \int_0^\infty \omega(\mathrm{d}\varphi^\rho(X)) \exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right)\mathrm{d}\rho - \omega(X)\Lambda(p)\ .$$

Proof. Let $\gamma : I \longrightarrow \mathcal{H}$ be curve such that $\gamma(0) = p$, $\gamma'(0) = X$. For every $\rho \geq 0$, the function

$$F_\rho : s \longmapsto \exp\left(\int_0^\rho \kappa(\varphi(\gamma(s), z))\mathrm{d}z\right)$$

is $\mathscr{C}^1$, with

$$F_\rho'(0) = \left(\int_0^\rho \frac{\partial\kappa(\varphi(\gamma(s), z))}{\partial s}\mathrm{d}z\right)\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right)\ .$$

More precisely, we have

$$\frac{\partial\kappa(\varphi(\gamma(s), z))}{\partial s} = \mathrm{d}\kappa(\mathrm{d}\varphi^z(X))\ .$$

Let us show that

$$\mathrm{d}\kappa(\mathrm{d}\varphi^z(X)) = \frac{\mathrm{d}\omega(\mathrm{d}\varphi^z(X))}{\mathrm{d}z}\ . \tag{5.7}$$

As $\mathcal{L}_n[\mathrm{d}\varphi^z(X)] = 0$, we have

$$\frac{\mathrm{d}\omega(\mathrm{d}\varphi^z(X))}{\mathrm{d}z} = \mathcal{L}_n\left[\omega(\mathrm{d}\varphi^z(X))\right] = \mathcal{L}_n\omega(\mathrm{d}\varphi^z(X))\ .$$

Hence (5.7) by Lemma 5.6. Thus,

$$F_\rho'(0) = (\omega(\mathrm{d}\varphi^\rho(X)) - \omega(X))\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right)\ .$$

By Lemma 5.7, and as $\kappa < -K$, there is $C > 0$ and $\lambda \in \mathbb{R}$ such that

$$|F'_\rho(0)| \leq \left(C\left(\sqrt{g(X,X)}(1+\rho\|\kappa\|_{L^\infty(\mathcal{H})}) + |\lambda|\right) + \|\kappa\|_{L^\infty(\mathcal{H})}\right) e^{-K\rho} \ . \tag{5.8}$$

Extending X to a vector field on $\mathcal{H}$, (5.8) shows that there are constants $\mu, \nu > 0$ such that for every $p' \in \mathcal{H}$,

$$\partial_X \left[\exp\left(\int_0^\rho \kappa(\varphi(p',z))\mathrm{d}z\right)\right] \leq (\mu\rho + \nu)e^{-K\rho} \ .$$

This domination function of ρ is integrable on $[0, +\infty)$ and independant of p. Thus, a classic theorem for the differentiation under the integral sign shows that Λ is $\mathscr{C}^1$, and the formula announced

$$\mathrm{d}\Lambda_p(X) = \int_0^\infty \partial_X \left[\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right)\right]\mathrm{d}\rho = \int_0^\infty \omega(\mathrm{d}\varphi^\rho(X))\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right)\mathrm{d}\rho - \omega(X)\Lambda(p) \ .$$

□

Corollary 5.2

The function Λ is $\mathscr{C}^1$, smooth along the null generators of $\mathcal{H}$, and satisfies

$$1 + \kappa\Lambda + \partial_n\Lambda = 0 \ .$$

Proof. With Proposition 5.10 it is enough to check that $1 + \kappa\Lambda + \partial_n\Lambda = 0$. By Proposition 5.10, as $\mathrm{d}\varphi^\rho(n(p)) = n(\varphi(p,\rho))$,

$$\begin{aligned}
\partial_n\Lambda(p) &= \int_0^\infty \omega(\mathrm{d}\varphi^\rho(n))\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right)\mathrm{d}\rho - \omega(n(p))\Lambda(p) \\
&= \int_0^\infty \kappa(\varphi(p,\rho))\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right)\mathrm{d}\rho - \kappa(p)\Lambda(p) \\
&= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\rho}\left[\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right)\right]\mathrm{d}\rho - \kappa(p)\Lambda(p) \\
&= \exp\left(\int_0^{+\infty} \kappa(\varphi(p,z))\mathrm{d}z\right) - 1 - \kappa(p)\Lambda(p) \\
&= -1 - \kappa(p)\Lambda(p)
\end{aligned}$$

as wished. □

The following results will be used to show that Λ is actually $\mathscr{C}^2$.

Proposition 5.11

There is a smooth $(0,2)$-type tensor μ on $\mathcal{H}$ such that for every $X, Y \in T\mathcal{H}$,

$$R(n,X)Y = \mu(X,Y)n \ .$$

Moreover,

$$\mu(X,Y) - \mu(Y,X) = -\mathrm{d}\omega(X,Y) \ ,$$

$\mu(n,\cdot) = \mu(\cdot,n) = 0$, *and μ is independant of the null non-zero vector n chosen.*

Proof. For the existence of μ, it is enough to show that $R(n,X)Y$ is null. Let $W \in T\mathcal{H}$. By the classic symmetry $R_{abcd} = R_{cdab}$ of the curvature tensor,

$$g(W, R(n,X)Y) = g(X, R(Y,W)n) = g(X,n)\mathrm{d}\omega(Y,W) = 0 \ .$$

Thus $R(n,Y)X$ is null, hence proportional to n by Proposition 2.1. The equation

$$\mu(X,Y) - \mu(Y,X) = -\mathrm{d}\omega(X,Y)$$

is a simple consequence of the first Bianchi identity $R_{abcd}+R_{acdb}+R_{adbc} = 0$. The last affirmations come from the fact that $R(n,X)n = \mathrm{d}\omega(n,X)n = 0$ as ω is null-closed, and from the fact that the equation defining μ is invariant by the multiplication of n by a scalar. □

Let η be the tensor of type $(0,2)$ defined by, for $X,Y \in T\mathcal{H}$,

$$\eta(X,Y) = \nabla_X\omega(Y) + \omega(X)\omega(Y) + \mu(X,Y) \ .$$

Lemma 5.8

For every $X,Y \in T\mathcal{H}$,

$$(\mathcal{L}_n\nabla)(X)Y = \eta(X,Y)n \ .$$

Proof. The proof is based on results about the Lie derivative of the connection that can be found in [Yan55, p. 9]. It is defined as the $(2,1)$-type tensor

$$(\mathcal{L}_n\nabla)(X)Y := \mathcal{L}_n[\nabla_X Y] - \nabla_{\mathcal{L}_n X}Y - \nabla_X\mathcal{L}_n Y$$

and it satisfies

$$(\mathcal{L}_n\nabla)(X)Y = R(n,X)Y + i_Y\nabla_X\nabla n \tag{5.9}$$

where i_Y is the contraction with Y. The announced formula comes from (5.9) and the following calculation :

$$\begin{aligned} i_Y\nabla_X\nabla n &= i_Y\nabla_X[\omega\cdot n] \\ &= i_Y[(\nabla_X\omega)\cdot n + \omega\cdot\nabla_X n] \\ &= ((\nabla_X\omega)(Y) + \omega(X)\omega(Y))n \ . \end{aligned}$$

□

Proposition 5.12

The function Λ is $\mathscr{C}^2$ and satisfies, for $X,Y \in \Gamma\mathcal{H}$,

$$\begin{aligned} \nabla_X\nabla_Y\Lambda(p) = \int_0^\infty &[\nabla_{X^\rho}\omega(Y^\rho) + \omega(\nabla^\rho_X Y^\rho) + (\omega(X^\rho) - \omega(X))(\omega(Y^\rho) - \omega(Y))]\exp\left(\int_{\varphi_p}^\rho \kappa\right)\mathrm{d}\rho \\ &- (\nabla_X\omega(Y) + \omega(\nabla_X Y))\Lambda(p) \end{aligned}$$

where $X^\rho := \mathrm{d}\varphi^\rho(X(p))$, $Y^\rho := \mathrm{d}\varphi^\rho(Y(p))$.

Proof. Let $p \in \mathcal{H}$, and let X,Y be smooth vector fields on $\mathcal{H}$. We need to show that $\partial_Y\partial_X\Lambda$ exists and is continuous. We saw in Proposition 5.10 that Λ is $\mathscr{C}^1$ and that

$$\partial_X\Lambda(p) = \int_0^\infty \omega(\mathrm{d}\varphi^\rho(X(p)))\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right)\mathrm{d}\rho - \omega(X(p))\Lambda(p) \ .$$

The term $-\omega(X)\Lambda$ is $\mathscr{C}^1$, thus we only need to show that

$$\partial_Y\left[p\longmapsto \int_0^\infty \omega(\mathrm{d}\varphi^\rho(X(p)))\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right)\mathrm{d}\rho\right]$$

exists and is continuous. Define

$$F_\rho(p) = \omega(\mathrm{d}\varphi^\rho(X(p)))\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right) .$$

For every $\rho \geq 0$, F_ρ is $\mathscr{C}^1$, and we can compute

$$\partial_Y F_\rho(p) = \nabla_Y[p \longmapsto \omega(\mathrm{d}\varphi^\rho(X(p)))]\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right) + \omega(\mathrm{d}\varphi^\rho(X(p)))\nabla_Y\left[\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right)\right]$$

We saw in Proposition 5.10 that

$$\nabla_Y\left[\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right)\right] = (\omega(\mathrm{d}\varphi^\rho(Y(p))) - \omega(Y(p))\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right) .$$

Define $X^\rho = \mathrm{d}\varphi^\rho(X(p))$ and $Y^\rho = \mathrm{d}\varphi^\rho(Y(p))$. Then

$$\partial_Y F_\rho(p) = (\nabla_{Y^\rho}[\omega(X^\rho)] + \omega(X^\rho)(\omega(Y^\rho) - \omega(Y)))\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right) .$$

To show that Λ is $\mathscr{C}^2$, we only need to show that this expression is dominated by an integrable function independant of p in a neighbourhood. Lemma 5.7 shows that the term $\omega(X^\rho)(\omega(Y^\rho) - \omega(Y))$ is dominated by a polynomial in ρ with continuous (thus bounded) coefficients. We also know that

$$\exp\left(\int_0^\rho \kappa(\varphi(p,z))\mathrm{d}z\right) \leq e^{-K\rho}$$

with $K > 0$. Thus we only need to show that the term $\nabla_{Y^\rho}[\omega(X^\rho)]$ is dominated by a polynomial in ρ with coefficients continuous with p. Like in the demonstration of Lemma 5.7, write the splitting

$$\begin{aligned} X^\rho &= \lambda_X(\rho)Z + X_H^\rho \\ Y^\rho &= \lambda_Y(\rho)Z + Y_H^\rho \end{aligned}$$

where X_H^ρ, $Y_H^\rho \in H$. We saw that λ_X and λ_Y are bounded by affine functions of ρ with coefficients continuous with p. Moreover,

$$\nabla_{Y^\rho}[\omega(X^\rho)] = (\nabla_{Y^\rho}\omega)(X^\rho) + \omega(\nabla_{Y^\rho}X^\rho)$$

and we can expand

$$(\nabla_{Y^\rho}\omega)(X^\rho) = (\nabla_{Y_H^\rho}\omega)(X_H^\rho) + \lambda_Y(\rho)(\nabla_n\omega)(X^\rho) + \lambda_X(\rho)(\nabla_{Y_H^\rho}\omega)(n) + \lambda_X(\rho)\lambda_Y(\rho)(\nabla_n\omega)(n) .$$

The term $(\nabla_{Y_H^\rho}\omega)(X_H^\rho)$ is a bilinear form evaluated on (X_H^ρ, Y_H^ρ), thus by continuity and compactness of the unitary subbundle of $(H \oplus H,\ g_H \oplus g_H)$, there is a global constant $M > 0$ such that

$$|(\nabla_{Y_H^\rho}\omega)(X_H^\rho)| \leq M\sqrt{g(X_H^\rho, X_H^\rho)g(Y_H^\rho, Y_H^\rho)} = S\sqrt{g(X(p),X(p))g(Y(p),Y(p))} .$$

Likewise, the terms $(\nabla_n\omega)(X^\rho)$ and $(\nabla_{Y_H^\rho}\omega)(n)$ are one-forms evaluated on X_H^ρ and Y_H^ρ, thus are bounded by global constants times $\sqrt{g(X(p),X(p)}$ and $\sqrt{g(Y(p),Y(p)}$. Finally, the term $\lambda_X(\rho)\lambda_Y(\rho)(\nabla_n\omega)(n)$ is bounded by a quadratic function with coefficients continuous with p. Thus, we only need to take care of the term $\omega(\nabla_{Y^\rho}X^\rho)$.
Let $(e_\mu) = e_1, \ldots, e_{n-1}$ be an orthonormal basis of H_p. Define, for $\rho \geq 0$, $e_\mu(\rho) = \mathrm{d}\varphi^\rho(e_\mu)$. Then $(e_1(\rho), \ldots, e_{n-1}(\rho), n(\varphi_p(\rho)))$ is a basis of $T_{\varphi_p(\rho)}\mathcal{H}$. Thus we can write

$$\nabla_{Y^\rho}X^\rho = \sum_{\mu=1}^{n-1} x^\mu(\rho)e_\mu(\rho) + \lambda(\rho)n(\varphi(p,\rho)) . \tag{5.10}$$

As $\mathcal{L}_n g = 0$ by Lemma 5.1, we can compute

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left[g(e_\mu(\rho), e_\nu(\rho))\right] = g(\mathcal{L}_n e_\mu(\rho), e_\nu(\rho)) + g(e_\mu(\rho), \mathcal{L}_n e_\nu(\rho)) = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left[g(e_\mu(\rho), n(\varphi(p,\rho)))\right] = g(\mathcal{L}_n e_\mu(\rho), Z(\varphi(p,\rho))) + g(e_\mu(\rho), \mathcal{L}_n n(\varphi(p,\rho))) = 0$$

thus $g(e_\mu(\rho), e_\nu(\rho)) = \delta_{\mu\nu}$ and $g(e_\mu(\rho), n(\varphi_p(\rho))) = 0$. This shows that $x^\mu(\rho) = g\left(\nabla_{Y^\rho} X^\rho, e_\mu(\rho)\right)$. We can thus compute

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\rho} = g(\mathcal{L}_n[\nabla_{Y^\rho} X^\rho], e_\mu(\rho)) = 0$$

as Lemma 5.8 shows that $\mathcal{L}_n[\nabla_{Y^\rho} X^\rho] = (\mathcal{L}_n\nabla)(X^\rho)Y^\rho = \eta(X^\rho, Y^\rho)n$ is proportional to n, because $\mathcal{L}_n X^\rho = \mathcal{L}_n Y^\rho = 0$. Thus, x^μ is constant. Now, taking the Lie derivative of (5.10) gives

$$\mathcal{L}_n[\nabla_{Y^\rho} X^\rho] = \lambda'(\rho) n(\varphi_p(\rho))$$

thus, by Lemma 5.8,

$$\begin{aligned}\lambda'(\rho) &= n^*(\mathcal{L}_n[\nabla_{Y^\rho} X^\rho]) \\ &= \eta(X^\rho, Y^\rho) \\ &= \mu(X^\rho, Y^\rho) + (\nabla_{Y^\rho}\omega)(X^\rho) + \omega(X^\rho)\omega(Y^\rho)\,.\end{aligned}$$

Moreover, we saw that the term $(\nabla_{Y^\rho}\omega)(X^\rho) + \omega(X^\rho)\omega(Y^\rho)$ was bounded by a polynomial in ρ with coefficients continuous with p. As we saw in Proposition 5.11,

$$\mu(X^\rho, Y^\rho) = \mu(X_H^\rho, Y_H^\rho)$$

and the bilinear form μ is continuous, thus bounded on the unitary subbundle of $(H \oplus H,\, g_H \oplus g_H)$ by compactness. This shows that there is a global constant $S > 0$ such that

$$\mu(X_H^\rho, Y_H^\rho) \leq S\sqrt{g(X_H^\rho, X_H^\rho)g(Y_H^\rho, Y_H^\rho)} = S\sqrt{g(X(p), X(p))g(Y(p), Y(p))}\,.$$

This shows that λ', hence λ, by integrating, is bounded by a polynomial in ρ with continuous coefficients. We can now conclude that

$$\omega(\nabla_{Y^\rho} X^\rho) = \sum_{\mu=1}^{n-1} x^\mu \omega(e_\mu(\rho)) + \lambda(\rho)\kappa(\varphi(p,\rho))$$

is bounded by a polynomial in ρ with continuous coefficients, because λ is, because the x^μ are constants, because the terms $\omega(e_\mu(\rho))$ are bounded by affine functions of ρ, as shown in Lemma 5.7, and because κ is bounded. To conclude, a classic theorem for the differentiation under the integral sign shows that Λ is $\mathscr{C}^2$, and that we can calculate the second derivative of Λ by differentiating under the integral sign, which directly gives the announced formula thanks to Proposition 5.10. □

The following result shows a link between the tensor μ and the geometry of $\mathcal{H}$, and will be used to show that Λ satisfies an elliptic partial differential equation in the vacuum case.

Proposition 5.13

We have the identity

$$\mathcal{L}_n\mu = \frac{1}{2}\mathcal{L}_n \mathrm{Ric}|_{\mathcal{H}}\,.$$

Proof. Let $(e_i)_{2\leq i\leq n}$ be a local orthonormal basis of $\ker\omega_p$. By definition of the Ricci tensor, for $X,Y\in T_p\mathcal{H}$,

$$\begin{aligned}\mathrm{Ric}(X,Y) &= \mathrm{Tr}\begin{pmatrix} T_p\mathcal{M} & \longrightarrow & T_p\mathcal{M} \\ W & \longmapsto & R(W,Y)X \end{pmatrix}\\ &= -g(R(n,Y)X,N) - g(R(N,Y)X,n) + \sum_{i=2}^n g(R(e_i,Y)X,e_i)\\ &= -g(R(n,Y)X,N) - g(R(n,X)Y,N) + \sum_{i=2}^n g(R(e_i,Y)X,e_i)\\ &= \mu(X,Y) + \mu(Y,X) + \sum_{i=2}^n R(e_i,Y,X,e_i)\end{aligned}$$

using the symmetries of the curvature tensor R_{abcd}. Extend e_i and X,Y such that $[n,X]=[n,Y]=0$, and such that (e_i) remains an orthonormal basis of $\ker\omega$. It is enough that e_i satisfies the ODE

$$\mathcal{L}_n e_i = \mathcal{L}_n g(N,e_i)n\ .$$

Then, by Proposition 5.11 and Lemma 5.6, and as the exterior derivative and the Lie derivative commute,

$$\begin{aligned}\mathcal{L}_n\mu(X,Y) &= \mathcal{L}_n\mu(Y,X) + \mathrm{d}\mathcal{L}_n\omega(X,Y)\\ &= \mathcal{L}_n\mu(Y,X) + \mathrm{d}^2\kappa(X,Y)\\ &= \mathcal{L}_n\mu(Y,X)\ .\end{aligned}$$

because $d^2=0$. Moreover, as $\mathcal{L}_n e_i$ is proportional to n,

$$\mathcal{L}_n[R(e_i,Y,X,e_i)] = \mathcal{L}_n R(e_i,Y,X,e_i)\ .$$

Thus,

$$\mathcal{L}_n\mathrm{Ric}(X,Y) = 2\mathcal{L}_n\mu(X,Y) + \sum_{i=2}^n \mathcal{L}_n R(e_i,Y,X,e_i)\ . \tag{5.11}$$

Moreover, in local coordinates $(x^\mu)_{0\leq\mu\leq n}$ such that $N=\partial_0$, $n=\partial_1$ and $\partial_2,\dots,\partial_n\in T\mathcal{H}$, as shown in [LD69, Eqn. (2.6)], denoting $\alpha=\mathcal{L}_n g$,

$$\mathcal{L}_n R^\ell_{ijk} = \frac{g^{\ell m}}{2}[(\alpha_{km;i}+\alpha_{mi;k}-\alpha_{ki;m})_{;j} - (\alpha_{km;j}+\alpha_{mj;k}-\alpha_{kj;m})_{;i}]\ .$$

Thus,

$$\mathcal{L}_n R_{abcd} = (\mathcal{L}_n g_{a\ell})R^\ell_{bcd} + \frac{1}{2}(\alpha_{da;b}+\alpha_{ab;d}-\alpha_{db;a})_{;c} - \frac{1}{2}(\alpha_{da;c}+\alpha_{ac;d}-\alpha_{dc;a})_{;b}$$

Suppose that $a,b,c,d\geq 1$. Then for $\ell\geq 1$, by Lemma 5.1, $\mathcal{L}_n g_{a\ell}=0$. Thus,

$$(\mathcal{L}_n g_{a\ell})R^\ell_{bcd} = -(\mathcal{L}_n g_{a0})R^0_{bcd} = 0$$

because R^0_{bcd} is the coordinate on N of $R(\partial_b,\partial_c)\partial_d$ in the basis $(\partial_\mu)_{0\leq\mu\leq n}$, but $R(\partial_b,\partial_c)\partial_d\in T\mathcal{H}$ as $\mathcal{H}$ is totally geodesic. Moreover, each term of the form $\alpha_{ij;k;l}=\partial_k\partial_l[\mathcal{L}_n g(\partial_i,\partial_j)]$ is zero for $i,j,k,l\geq 1$ because $\mathcal{L}_n g=0$ on $T\mathcal{H}$ by Lemma 5.1. Thus, $\mathcal{L}_n R=0$ on $T\mathcal{H}$, which shows the result, by (5.11). ☐

Define B the tensor of type $(0,2)$ by, for $X,Y\in T\mathcal{H}$,

$$B(X,Y) = -\int_0^\infty (\mu(X^\rho,Y^\rho)-\mu(X,Y))\exp\left(\int_{\varphi_p}^\rho \kappa\right)\mathrm{d}\rho\ .$$

Proposition 5.14

The function Λ *is smooth and satisfies the partial differential equation*

$$\operatorname{Hess}\Lambda + \omega \otimes \mathrm{d}\Lambda + \mathrm{d}\Lambda \otimes \omega + \eta \otimes \Lambda = B \ .$$

Moreover, if the spacetime is vacuum, this expression equals zero.

In other words, for $X, Y \in T\mathcal{H}$,

$$\operatorname{Hess}\Lambda(X,Y) + \omega(X)\nabla_Y\Lambda + \omega(Y)\nabla_X\Lambda + \eta(X,Y)\Lambda = B(X,Y) \ .$$

The smoothness is proved by induction in Appendix C, thus we only prove here the announced formula and the last statement.

Proof. Recall that $\operatorname{Hess}\Lambda := \nabla\mathrm{d}\Lambda$. With Propositions 5.12 and 5.10, we can compute

$$\begin{aligned}
\operatorname{Hess}\Lambda(X,Y) &= \nabla_X\nabla_Y\Lambda - \nabla_{\nabla_X Y}\Lambda \\
&= \int_0^\infty \big[\nabla_{X^\rho}\omega(Y^\rho) - \nabla_X\omega(Y) \\
&\quad + \omega(\nabla_{X^\rho}Y^\rho) - \omega((\nabla_X Y)^\rho) \\
&\quad + (\omega(X^\rho) - \omega(X))(\omega(Y^\rho) - \omega(Y))\big] \exp\left(\int_{\varphi_p}^\rho \kappa\right) \mathrm{d}\rho \ .
\end{aligned}$$

We will show that

$$\omega(\nabla_{X^\rho}Y^\rho) - \omega((\nabla_X Y)^\rho) = \kappa \int_0^\rho \mu(X^s, Y^s) + \nabla_{X^s}\omega(Y^s) + \omega(X^s)\omega(Y^s)\mathrm{d}s \ . \tag{5.12}$$

Lemma 5.8 shows that

$$\begin{aligned}
\mathcal{L}_n[\nabla_{X^\rho}Y^\rho - (\nabla_X Y)^\rho] &= \eta(X^\rho, Y^\rho)n \\
&= \mathcal{L}_n\left[n \int_0^\rho \mu(X^s, Y^s) + \nabla_{X^s}\omega(Y^s) + \omega(X^s)\omega(Y^s)\mathrm{d}s\right]
\end{aligned}$$

Moreover,

$$\nabla_{X^0}Y^0 - (\nabla_X Y)^0 = 0 = \left[\int_0^0 \mu(X^s, Y^s) + \nabla_{X^s}\omega(Y^s) + \omega(X^s)\omega(Y^s)\mathrm{d}s\right] n \ .$$

Thus, the uniqueness of the solution of the first order ODE $\mathcal{L}_n Z(\rho) = 0$, $Z(0) = 0$ shows that

$$\nabla_{X^\rho}Y^\rho - (\nabla_X Y)^\rho = \left[\int_0^\rho \mu(X^s, Y^s) + \nabla_{X^s}\omega(Y^s) + \omega(X^s)\omega(Y^s)\mathrm{d}s\right] n$$

hence (5.12) taking ω of the previous equation. Using this fact, and expanding the product, we get

$$\begin{aligned}
\operatorname{Hess}\Lambda(X,Y) &= \int_0^\infty \left[\nabla_{X^\rho}\omega(Y^\rho) + \omega(X^\rho)\omega(Y^\rho) + \kappa\int_0^\rho \nabla_{X^s}\omega(Y^s) + \omega(X^s)\omega(Y^s)\mathrm{d}s\right]\exp\left(\int_{\varphi_p}^\rho \kappa\right)\mathrm{d}\rho \\
&\quad - \omega(X)\int_0^\infty \omega(Y^\rho)\exp\left(\int_{\varphi_p}^\rho \kappa\right)\mathrm{d}\rho - \omega(Y)\int_0^\infty \omega(X^\rho)\exp\left(\int_{\varphi_p}^\rho \kappa\right)\mathrm{d}\rho \\
&\quad + \int_0^\infty \kappa\left[\int_0^\rho \mu(X^s,Y^s)\mathrm{d}s\right]\exp\left(\int_{\varphi_p}^\rho \kappa\right)\mathrm{d}\rho \\
&\quad + (\omega(X)\omega(Y) - \nabla_X\omega(Y))\Lambda \\
&= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}\rho}\left[\left(\int_0^\rho \nabla_{X^s}\omega(Y^s) + \omega(X^s)\omega(Y^s)\mathrm{d}s\right)\exp\left(\int_{\varphi_p}^\rho \kappa\right)\right]\mathrm{d}\rho \\
&\quad - \omega(X)(\nabla_Y\Lambda + \omega(Y)\Lambda) - \omega(Y)(\nabla_X\Lambda + \omega(X)\Lambda) \\
&\quad - \int_0^\infty \mu(X^\rho,Y^\rho)\exp\left(\int_{\varphi_p}^\rho \kappa\right)\mathrm{d}\rho + (\omega(X)\omega(Y) - \nabla_X\omega(Y))\Lambda \\
&= -\int_0^\infty [\mu(X^\rho,Y^\rho) - \mu(X,Y)]\exp\left(\int_{\varphi_p}^\rho \kappa\right)\mathrm{d}\rho - \eta(X,Y)\Lambda - \omega(X)\nabla_Y\Lambda - \omega(Y)\nabla_X\Lambda
\end{aligned}$$

recognizing a total derivative and integrating by parts. We have proved the announced formula. Suppose now that the spacetime is vacuum, we need to prove that

$$\int_0^\infty [\mu(X^\rho,Y^\rho) - \mu(X,Y)]\exp\left(\int_{\varphi_p}^\rho \kappa\right)\mathrm{d}\rho = 0\,.$$

This is because in vacuum, the stress energy-tensor T is zero, thus, the Einstein equation (4.1) implies $\mathrm{Ric} = 2\lambda/(n-1)g$ where λ is the cosmological constant. We conclude by Proposition 5.13 and Lemma 5.1 that

$$\frac{\mathrm{d}}{\mathrm{d}\rho}[\mu(X^\rho,Y^\rho)] = \mathcal{L}_n\mu(X^\rho,Y^\rho) = 0\,.$$

□

Notice that in the vacuum case, and actually in any case where $\mathcal{L}_n\mathrm{Ric}|_{\mathcal{H}} = 0$, the elliptic PDE

$$\operatorname{Hess}\Lambda + \omega\otimes\mathrm{d}\Lambda + \mathrm{d}\Lambda\otimes\omega + \eta\otimes\Lambda = 0$$

directly proves the smoothness of Λ by a bootstrap argument.

Theorem 5.1

The vector field $h := \Lambda n$ on $\mathcal{H}$ is smooth, null, nowhere-zero and satisfies $\nabla_h h = -h$. The null generators starting with tangent h have an affine length equal to 1.

Notice that h doesn't depend on the chosen null vector field n. Recall that this results holds assuming only the dominant energy condition. This vector field h can be interpreted as some sort of « homogeneity » vector field, in the sense that h normalises to the constant 1 the affine lengths Λ' of the lightlike geodesics of $\mathcal{H}$ emanating with initial velocity equal to h, which is equivalent to normalizing the surface gravity $\kappa' = \omega_h(h)$ to -1.

Proof. The fact that h is smooth is a consequence of Proposition 5.14. Moreover, $\nabla_h h = -h$ because $\omega_h(h) = -1$ by Proposition 5.6 and Corollary 5.2. The last affirmation was proved in Section 3.3. □

Recall that we denote $\mathcal{N} \longrightarrow \mathcal{H}$ the null bundle of $\mathcal{H}$. Some geometric consequences of the results of this section can be stated as follows :

Proposition 5.15

Let h be the homogeneity vector field on $\mathcal{H}$ given by Theorem 5.1, and redefine $\omega := \omega_h$. Then :

$$\mathcal{L}_h g = 0 \text{ on } T\mathcal{M}|_{\mathcal{H}}$$

$$\mathcal{L}_h R = 0 \text{ on } T\mathcal{H} .$$

Moreover, if the spacetime is vacuum, the Riemann curvature tensor is characterized on $\mathcal{H}$ by ω in the following sense :

$$R(X,Y)Z = \mathrm{d}\omega(X,Y)Z$$

$$R(X,Z)Y = (\nabla\omega(X,Y) + \omega(X)\omega(Y))\, Z$$

for $X, Y \in T\mathcal{H}$ and $Z \in \mathcal{N}$.

Proof. We use the same notations for κ, η and N as in this section, but defined for $n = h$. With Lemma 5.1, we only need to show that for $X \in T\mathcal{H}$,

$$\mathcal{L}_h g(X,N) = \mathcal{L}_h g(N,N) = 0 .$$

By definition of N and linearity, $\omega = N^\flat$. Moreover, Lemma 5.6 shows that

$$\mathcal{L}_h \omega = \mathrm{d}\kappa = 0$$

because $\kappa = -1$. Thus, on $T\mathcal{H}$,

$$0 = \mathcal{L}_h g(N, \cdot\,) + g(\mathcal{L}_h N, \cdot) = \mathcal{L}_h g(N, \cdot\,)$$

because we extended h such that $[h, N] = \mathcal{L}_h N = 0$. Moreover, as $g(N,N) = 0$,

$$0 = \mathcal{L}_h[g(N,N)] = \mathcal{L}_h g(N,N) + 2g(\mathcal{L}_h N, N) = \mathcal{L}_h g(N,N)$$

hence the fact that $\mathcal{L}_h g = 0$ on $T\mathcal{M}|_{\mathcal{H}}$. We already proved the second and third facts. Proposition 5.14 (with $B = 0$ in the vacuum case) applied to $n = h$ reads, since $\Lambda = 1$, $\eta = 0$. Thus $\mu = -\nabla\omega - \omega \otimes \omega$, hence the fourth fact. □

As explained in Section 3, applying [PR18] and [Pet19], a corollary of Theorem 5.1 is that, in the vacuum case, h can be extended as a Killing vector field to a neighbourhood of $\mathcal{H}$, that is spacelike in the totally hyperbolic region and timelike on the other side. In other words, in a vacuum spacetime, any non-degenerate compact Cauchy horizon is a Killing horizon.

6 Conclusion

Conclusion and perspectives

To conclude the mathematical content of this report, notice first that the results obtained hold for any compact connected smooth totally geodesic null hypersurface $\mathcal{H}$. Indeed, the fact the $\mathcal{H}$ is a Cauchy horizon was only used in Theorem 2.1, and the conclusion of this theorem is still true for such a hypersurface (see [Min15]). This fact also shows that the connectedness of $\mathcal{H}$ is not a necessary hypothesis, as the results can be applied to each of its connected components that satisfies the other hypotheses. Thus, the existence of a homogeneity vector field as in Theorem 5.1 and Proposition 5.15 holds for more general hypersurfaces such as compact null boundaries of chronology violating sets, that are not necessarily Cauchy horizons.

On the other hand, the hypothesis of non-degeneracy cannot be dropped in our way of proceeding. Thus, it would be interesting to study whether there exist degenerate compact Cauchy horizons. As illustrated by the Isenberg-Moncrief conjecture, it is suspected that it is not the case. Indeed, if this conjecture is true, any compact Cauchy horizon is a Killing horizon, thus has a constant surface gravity as shown in [Chr20] : the associated Killing field K normalises the surface gravity to a constant κ, *i.e.* $\nabla_K K = \kappa K$. Then, unless $\kappa = 0$, the horizon is non-degenerate.

Appendix D introduces a way of characterizing potentially degenerate horizons by studying the behavior of the integral of the connection form on the generators (we saw that it converges to $-\infty$ in the non-degenerate case). We prove that it cannot diverge to $+\infty$ and that, under disputable assumptions stated in [Haw92], it cannot oscillate without being bounded. Thus, it might be interesting to study such «bounded horizons», and try to find nice properties that they satisfy. In particular, notice that in the vanishing surface gravity case $\kappa = 0$ stated above, the horizon is bounded in the previous sense. It might be interesting to study whether the converse is true, *i.e.* if every bounded horizon has vanishing surface gravity (if moreover the generators are assumed to be closed, this is shown in [MI83]). Using the Poincaré recurrence theorem, we also prove in this appendix that in the non-degenerate case, every generator almost closes, with consequence the fact that a compact Cauchy horizon is in the closure of the chronology violating set.

Conclusion générale

J'ai beaucoup apprécié ce stage. J'ai trouvé le sujet passionnant à la fois au niveau des significations physiques des objets que je manipulais, qu'au niveau du contenu mathématique qui est très géométrique, comme souhaité. J'ai aussi pu assister au séminaire en ligne SCRI21 organisé par mon encadrant Ettore Minguzzi, durant lequel j'ai suivi des présentations de chercheurs de toutes nationalités, dont Roger Penrose.

Niveau travail, je n'ai pas échangé avec beaucoup de gens de l'université, ce qui m'a « forcé » à travailler souvent en solitaire, bien que nous faisions le point deux fois par semaine avec mon encadrant. Je ne l'ai pas mal vécu, car j'ai eu la chance de ne jamais être vraiment bloqué et d'être toujours motivé. Je crois quand même être passé plusieurs fois par l'ascenseur émotionnel du chercheur : fausse découverte, désillusion, re-travail, avant d'obtenir enfin un résultat solide.

Au final, j'ai été conforté dans mon envie de poursuivre une carrière de chercheur ou d'enseignant-chercheur. Le sujet sur lequel j'ai travaillé m'a beaucoup fait progresser et confirme mon goût pour la géométrie différentielle. Je pense donc continuer à me spécialiser dans ce domaine, de préférence en interaction avec la physique. Ce stage à aussi été l'occasion de pouvoir profiter de Florence, de visiter d'autres paysages de Toscane, ainsi que Rome, et de rencontrer hors de l'université beaucoup de gens différents de ce dont j'ai l'habitude. Mais par dessus tout, ce stage m'a apporté beaucoup de pizzas.

A Semi-Riemannian geometry

This section presents the results of semi-Riemannian geometry that are most important in this report. See [ONe83] for reference.

A.1 Fundamental aspects

The manifolds considered here are assumed to be Haussdorff and second countable (that is, a topological space that has a countable basis for its topology). Let $\mathcal{M}$ be a smooth manifold. Denote $T^*\mathcal{M} \otimes_{\mathcal{M}} T^*\mathcal{M}$ the set of the pairs (p, η) with $p \in \mathcal{M}$ and η a symmetric bilinear form on $T_p\mathcal{M}$, and $\Gamma\mathcal{M}$ the set of smooth vector fields on $\mathcal{M}$.

Definition A.1

A $\mathscr{C}^k$ metric g on $\mathcal{M}$ is a $\mathscr{C}^k$ section of the bundle $T^*\mathcal{M} \otimes_{\mathcal{M}} T^*\mathcal{M} \longrightarrow \mathcal{M}$. That is, a $\mathscr{C}^k$ metric is an application $g : \mathcal{M} \longrightarrow T^*\mathcal{M} \otimes_{\mathcal{M}} T^*\mathcal{M}$ such that for every $p \in \mathcal{M}$, g_p is a symmetric bilinear form on $T_p\mathcal{M}$, and such that for every smooth vector fields $X, Y \in \Gamma\mathcal{M}$, the function

$$g(X, Y) : p \in \mathcal{M} \longmapsto g_p(X(p), Y(p)) \text{ is } \mathscr{C}^k .$$

If at every point the metric g is positive definite, g is said to be a *Riemannian metric*, and $(\mathcal{M}, g)$ is a *Riemannian manifold*.

The metric g is said to be non-degenerate if it is non-degenerate at every point $p \in \mathcal{M}$, *i.e.* if there is no non-zero vector $X \in T_p\mathcal{M}$ such that the linear form $g_p(X, \, . \,)$ is zero. Equivalently, g is non-degenerate if at every point the matrix of g in any basis is invertible.

Definition A.2

A semi-Riemannian manifold is a pair $(\mathcal{M}, g)$ where $\mathcal{M}$ is a smooth manifold and where g is a non-degenerate smooth metric on $\mathcal{M}$.

Theorem A.1

On a semi-Riemannian manifold $(\mathcal{M}, g)$, there is a unique torsion-free connection ∇ on $T\mathcal{M}$ that preserves the metric, i.e. such that $\nabla g = 0$.

This special connection is called the *Levi-Civita connection* of $(\mathcal{M}, g)$. The torsion-less property means that for every $X, Y \in \Gamma\mathcal{M}$, $[X, Y] = \nabla_X Y - \nabla_Y X$.

Recall that $\Gamma\mathcal{M}$ is the set of vector fields on $\mathcal{M}$. Denote $\Gamma^*\mathcal{M}$ the set of one-forms on $\mathcal{M}$.

Proposition A.1

The map

$$\begin{array}{lll} \Gamma\mathcal{M} & \longrightarrow & \Gamma^*\mathcal{M} \\ X & \longmapsto & X^\flat = g(X, \cdot\,) \end{array}$$

is an isomorphism, called the musical isomorphism.

The inverse of the musical isomorphism is denoted

$$\begin{array}{lll} \Gamma^*\mathcal{M} & \longrightarrow & \Gamma\mathcal{M} \\ \omega & \longmapsto & \omega^\sharp \end{array} .$$

The notion of curvature in semi-Riemannian geometry is dealt through the following tensor field, which measures how covariant derivatives of vector fields fail to commute (which is a property of *non-flat* spaces) :

Definition A.3

The Riemann curvature tensor of $(\mathcal{M}, g)$ *is the tensor of type* $(3,1)$ *defined by, for* $X, Y, Z \in T\mathcal{C}$,

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \ .$$

Notice that this definition means that R is the $(3,1)$ tensor such that for $\omega \in T^*\mathcal{C}$,

$$R(X,Y,Z,\omega) = \omega(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z)$$

but this notation is avoided. Indeed, it is used for the purely covariant version $R_{abcd} = g_{a\ell} R^{\ell}_{bcd}$ of the curvature tensor, that satisfies

$$R(W,X,Y,Z) = g(R(W,X)Y,Z) \ .$$

Definition A.4

The Ricci tensor of $(\mathcal{M}, g)$ *is the tensor of type* $(2,10)$ *defined by, for* $X, Y \in T\mathcal{C}$,

$$\mathrm{Ric}(X,Y) = \mathrm{Tr} \begin{pmatrix} T\mathcal{M} & \longrightarrow & T\mathcal{M} \\ Z & \longmapsto & R(Z,Y)X \end{pmatrix} \ .$$

The scalar curvature is the function $R : \mathcal{M} \longrightarrow \mathbb{R}$ *defined in coordinates as*

$$R = \mathrm{Tr}(g^{-1}\mathrm{Ric}) = g^{\mu\nu} R_{\mu\nu}$$

where $R_{\mu\nu}$ *is the coordinates of* Ric.

Denote d the exterior derivative.

Proposition A.2

Let ω *be a one-form on* $\mathcal{M}$. *Then for* $X, Y \in T\mathcal{C}$,

$$\mathrm{d}\omega(X,Y) = \nabla_X[\omega(Y)] - \nabla_Y[\omega(X)] - \omega([X,Y]) \ .$$

Definition A.5

Let $f : \mathcal{M} \longrightarrow \mathbb{R}$ *be a* $\mathscr{C}^2$ *function. The Hessian of* f *is defined as* $\mathrm{Hess}\, f = \nabla \mathrm{d} f$. *More precisely, it is the tensor field*

$$\mathrm{Hess}\, f(X,Y) = \nabla_X \nabla_Y f - \nabla_{\nabla_X Y} f \ .$$

Definition A.6

A curve γ *on* $\mathcal{M}$ *is called a geodesic if it satisfies the geodesic equation* $\nabla_{\gamma'} \gamma' = 0$.

A curve $\gamma : (a,b) \longrightarrow \mathcal{M}$ is said to be past-inextendible (resp. future-inextendible) if $\gamma(t)$ doesn't converge when $t \to a$ (resp. when $t \to b$). The geodesic equation is a linear second-order ODE. Thus, for every $p \in \mathcal{M}$ and $X \in T_p\mathcal{M}$, there is a unique inextendible geodesic γ defined on a maximal interval $[0, L)$ such that $\gamma(0) = p$ and $\gamma'(0) = X$. The number L is called *the affine length of* γ *starting from* p and it can be finite or infinite. If it is finite, γ is said to be *incomplete*, and *complete* otherwise.

The Lie derivative $\mathcal{L}$ is a tensor derivation along vector fields. It can be defined by induction on the type of the tensor field. For vector fields $X, Y \in \Gamma\mathcal{M}$, denoting $[\cdot\,,\cdot]$ the Lie bracket, it satisfies

$$\mathcal{L}_X Y = [X,Y] = \nabla_X Y - \nabla_Y X \ .$$

For functions $f : \mathcal{M} \longrightarrow \mathbb{R}$, it satisfies

$$\mathcal{L}_X f = \partial_X f \ .$$

Finally, for covariant tensors T, it satisfies

$$\mathcal{L}_X T(p) = \frac{\mathrm{d}}{\mathrm{d}t} \left[(\varphi_X^t)^* (T(\varphi_X^t(p)) \right]$$

where $(\varphi_X^t)^*$ is the pullback by the flow φ_X^t of X.

Definition A.7

A vector field $X \in \Gamma\mathcal{M}$ is called a Killing field if

$$\mathcal{L}_X g = 0 \ .$$

It can be shown that $X \in \Gamma\mathcal{M}$ is a Killing field if and only if the flows φ_X^t of X are local isometries. Thus, a Killing field is a vector field that preserves the metric along its flow. It can also be shown that a Killing field also preserves the Riemann curvature tensor and the scalar curvature along its flow. The existence of a Killing field on $\mathcal{M}$ is a special property that implies a strong symmetry of $\mathcal{M}$.

A.2 Some properties of fiber bundles

Proposition A.3

Let $p : H \longrightarrow \mathcal{M}$ be a vector bundle on a compact manifold $\mathcal{M}$ with a positive definite fiber metric g. The unitary subbundle defined as

$$\mathscr{U} = \left\{ (p,v) \in H \;\middle|\; g(v,v) = 1 \right\}$$

is compact.

Proof. For any trivializing open set $D \subset \mathcal{M}$, as $\mathcal{M}$ is locally compact, there is a trivializing open set $\mathcal{V} \hookrightarrow D$ such $\overline{\mathcal{V}}$ is compact. Thus, as $\mathcal{M}$ is compact, there are such open sets $\mathcal{V}_i \hookrightarrow D_i$, for $1 \leq i \leq m$, such that $\mathcal{M} = \cup_{i=1}^m \mathcal{V}_i$. Denote by ϕ_i the homeomorphism defining D_i,

$$\begin{array}{rccc} \phi_i : & D_i \times \mathbb{R}^n & \longrightarrow & H \cap \mathrm{pr}_1^{-1}(D_i) \\ & (p, \mu_1, \dots, \mu_n) & \longmapsto & (p, \sum_{k=1}^n \mu_k v_k(p)) \end{array} .$$

Thanks to the Gram-Schmidt algorithm for vector bundles, we can suppose that the v_i's are orthonormals.
Now, if $(p,v) \in \mathscr{U}$, and if i is such that $p \in \mathcal{V}_i$, if $(\mu_1, \dots, \mu_n) = \mathrm{pr}_2(\phi_i^{-1}(p,v))$, we have

$$v = \sum_{k=1}^n \mu_k v_k(p)$$

thus $1 = \|v\|^2 = \sum_{i=1}^n \mu_k^2$. This an equivalence, so

$$\mathscr{U} \cap \mathrm{pr}_1^{-1}(\overline{\mathcal{V}_i}) = \phi_i^{-1}(\overline{\mathcal{V}_i} \times \mathbb{S}_{\mathbb{R}^n}(0,1))$$

is compact. Finally,

$$\mathscr{U} = \bigcup_{i=1}^n \mathscr{U} \cap \overline{\mathcal{V}_i}$$

is compact as a finite union of compact sets. □

Corollary A.1

Let $p : H \longrightarrow \mathcal{M}$ be a vector bundle on a compact manifold $\mathcal{M}$ with a positive definite fiber metric g. The frame bundle associated to H, defined as

$$\mathscr{F} = \left\{ (p, e_1, \dots, e_n) \,\middle|\, p \in \mathcal{M},\ e_i \in H_p,\ (e_1, \dots, e_n) \text{ is an orthonormal basis of } H_p \right\}$$

is compact.

Proof. With the notations of Proposition A.3, we clearly have $\mathcal{F} \subset \mathscr{U} \times_{\mathcal{M}} \dots \times_{\mathcal{M}} \mathscr{U}$ (fiber product over $\mathcal{M}$, equal to the tuples $(p, u_1, \dots, u_1)$ where u_i is in the unit sphere of H_p) which is compact thanks to Proposition A.3, because it is a closed subset of the compact space $\mathscr{U}^n$. Thus, we only need to show that $\mathcal{F}$ is closed in $\mathscr{U} \times_{\mathcal{M}} \dots \times_{\mathcal{M}} \mathscr{U}$. It is clear because for $(p, v_1, \dots, v_n) \in \mathscr{U} \times_{\mathcal{M}} \dots \times_{\mathcal{M}} \mathscr{U}$, we have

$$(p, v_1, \dots, v_n) \in \mathcal{F} \iff \forall i \neq j,\ g(v_i, v_j) = 0$$

and the functions $(p, v_1, \dots, v_n) \longmapsto g(v_i, v_j)$ are continous, hence the result. □

Proposition A.4

Let $\mathcal{M}$ be a smooth manifold and let ω be a smooth nowhere-zero one-form on $\mathcal{M}$. Then $\ker \omega \longrightarrow \mathcal{M}$ is a smooth subbundle of $T\mathcal{M}$.

Proof. This result is a direct corollary of [Lee83, Th 10.34], which states that the image and kernel of a constant rank smooth map are subbundles, in the case where the rank of the map is constant to one. □

Proposition A.5

Let $\pi : E \longrightarrow \mathcal{M}$ be a smooth vector bundle and let $F \longrightarrow \mathcal{M}$, $G \longrightarrow \mathcal{M}$ be smooth subbundles of π such that $\dim(F_p \cap G_p)$ is independant of $p \in \mathcal{M}$. Then $F \cap G$ is a smooth subbundle of π.

Proof. We know that $E \oplus E \longrightarrow \mathcal{M}$ is a smooth bundle for which $F \oplus G$ is a smooth subbundle, and that admits a smooth map

$$D : \begin{array}{ccc} E \oplus E & \longrightarrow & E \\ (v, w) & \longmapsto & v - w \end{array}.$$

Define $\widetilde{D} = D|_{F \oplus G}$. For $p \in \mathcal{M}$, $\ker \widetilde{D}_p = (F_p \cap G_p)^2$ has a dimension independant of p. Thus, $\widetilde{D}$ satisfies the hypothesis of [Lee83, Th 10.34], which shows that $\ker \widetilde{D}$ is a smooth subbundle of $E \oplus E \longrightarrow \mathcal{M}$. Hence the result, as $F \cap G$ can be identified with $\ker \widetilde{D}$. □

A.3 Existence and uniqueness of horizontal geodesics

The proof of Proposition A.2 is based on the construction of a suitable neighbourhood of $p \in \mathcal{H}$ (a foliated cylinder) on which horizontal geodesics will be in association with the usual geodesics of a specific quotient Riemannian manifold.

Construction of a suitable neighbourhood

Let $X \in \Gamma\mathcal{H}$ be a null nowhere zero vector field on $\mathcal{H}$, and denote its flow by $\varphi : \mathcal{H} \times \mathbb{R} \longrightarrow \mathcal{H}$. We will denote the same way φ_t and $\varphi(.\,, t)$. Let $(\mathcal{W}, \psi)$ be a chart containing p, with $\psi(p) = 0$, $\psi : \mathcal{W} \longrightarrow \mathcal{V} \hookleftarrow\!\!\!\rightarrow \mathbb{R}^n$. Let $\varepsilon > 0$ and denote $N_p = \mathrm{d}\psi_p(X(p))$, $W_p = N_p^{\perp \mathbb{R}^n}$, $D_0 = \mathsf{B}_{\mathbb{R}^n}(0, \varepsilon) \cap W_p$ and finally $D = \psi^{-1}(D_0)$.

For $q \in D$, denote as well $N_q = \mathrm{d}\psi_q(X_q)$. As $T_{\psi(p)}D_0 = W_p$, D_0 is transversal to $\mathbb{R}N_p$, which shows that D is transversal in p to $\mathbb{R}X(p)$, as $T_{\psi(p)}D_0 = \mathrm{d}\psi_p(T_pD)$ and $\mathbb{R}N_p = \mathrm{d}\psi_p(\mathbb{R}X(p))$. Moreover, D remains transversal to X on all of D for $\varepsilon > 0$ small enough.
Indeed, the map $q \in D \longmapsto N_q$ is continuous and by the definition of D, for every $q \in D$, $\mathrm{d}\psi_q T_q D = T_{\psi(q)}D_0 = W_p$, and $N_p \in W_p^c \underset{\circ}{\hookrightarrow} \mathbb{R}^n$, which shows that for $\varepsilon > 0$ small enough, for every $q \in D$, $N_q \in W_p^c$, *i.e.* $N_q \notin \mathrm{d}\psi_q T_q D$, which means that the hyperplane $\mathrm{d}\psi_q T_q D$ is transversal to N_q, and thus that D is transversal to X.
As $\overline{D} = \psi^{-1}(\overline{\mathsf{B}_{\mathbb{R}^n}(0,\varepsilon) \cap W_p})$ is compact, there is $\delta > 0$ such that the flow φ of X is defined on $D \times (-\delta, \delta)$. Moreover,

$$\mathrm{d}\varphi_{(p,0)}(Y, s) = sX(p) + Y$$

so $\mathrm{d}\varphi_{(p,0)} : T_{(p,0)}D \times (-\delta,\delta) \longrightarrow T_p\mathcal{H}$ is surjective, as D is transversal to X, hence an isomorphism as the dimensions are equal. By the Inverse function theorem, this shows that for $\delta > 0$ small enough, and by reducing D if necessary, $\varphi : D \times (-\delta,\delta) \longrightarrow \mathcal{H}$ is a diffeomorphisme on its image $\mathcal{U} = \varphi(D \times (-\delta,\delta)) \underset{\circ}{\hookrightarrow} \mathcal{H}$.
Note that $\mathcal{U}$ is foliated with the curves $C_x = \varphi(\{x\} \times (-\delta,\delta))$ for $x \in C_x$. Introduce the equivalence relation on $\mathcal{U}$ defined by

$$a \sim b \quad \text{if} \quad \exists x \in D,\ a, b \in C_x$$

and consider the quotient $V := \mathcal{U}/\sim$ (with the quotient topology), and the canonical projection $\xi : \mathcal{U} \longrightarrow V$. The map

$$\begin{array}{rccc} \xi|_D : & D & \longrightarrow & V \\ & x & \longmapsto & \xi(x) \end{array}$$

is a homeomorphism. Indeed, by definition of $\mathcal{U}$ and $\sim$, it is continuous and surjective, and it is injective as for every $x \in D$, $C_x \cap D = \{x\}$ by injectivity of φ. Thus, it is a continuous bijective map, and it is also open because if $\mathcal{V} \underset{\circ}{\hookrightarrow} D$, $\xi^{-1}(\xi(\mathcal{V})) = \varphi(\mathcal{V} \times (-\delta,\delta)) \underset{\circ}{\hookrightarrow} \mathcal{U}$ as φ is a diffeomorphism, so $\xi(\mathcal{V}) \underset{\circ}{\hookrightarrow} V$ by definition of the quotient topology. This show that $\xi|_D$ is a homeomorphism.
Now, consider the smooth manifold structure induced by D and $\xi|_D$ on V, *i.e.* such that $\xi|_D$ is a diffeomorphism. Denote for $|t| < \delta$, $D_t := \varphi(D \times \{t\})$. As φ_t is a diffeomorphism, $D_t = \varphi_t(D)$ is a smooth hypersurface of $\mathcal{U}$.
Denote also $\xi_t := \xi|_{D_t} : D_t \longrightarrow V$. As $\xi_t = \xi|_D \circ \varphi_t|_{D_t}$, and as $\xi|_D$ and $\varphi_t|_{D_t}$ are diffeomorphisms on their images, ξ_t is a diffeomorphism.
Moreover, as each φ_t is a diffeomorphism, and as $\mathrm{d}(\varphi_t)_x(X_x) = X(\varphi_t(x))$ (the flow preserves X), each D_t is still transversal to X because for $x \in D$,

$$T_{\varphi_t(x)}D_t + \mathbb{R}X_{\varphi_t(x)} = \mathrm{d}(\varphi_t)_x(T_xD + \mathbb{R}X_x) = T_{\varphi_t(x)}\mathcal{H} \ .$$

Now, if $f \in \mathcal{F}(V)$, denote $f^* = f \circ \xi \in \mathcal{F}(\mathcal{U})$. Recall that we denote $\pi : T\mathcal{H} = \mathcal{N} \oplus H \longrightarrow H$ the canonical projection. Given a vector field $Y \in \Gamma V$, we can define a smooth horizontal lift $Y^* \in \Gamma\mathcal{U}$ of Y as follows : for $p \in D_t$,

$$Y^*(p) = \pi\left(\left(\mathrm{d}(\xi_t)_p\right)^{-1} Y(\xi(p)\right) \in H_p$$

(Recall that $\xi_t : D_t \longrightarrow V$ is a diffeomorphism.) Note that by definition of π,

$$Y^*(p) \in \left(\mathrm{d}(\xi_t)_p\right)^{-1} Y(\xi(p) + \mathbb{R}X(p) \tag{A.1}$$

and for $f \in \mathcal{F}(V)$,

$$\mathrm{d}\xi_p(X(p))f = X(f \circ \xi)_p = (\mathcal{L}_X f \circ \xi)_p = 0$$

because $f \circ \xi$ is constant along the flow of X by definition of ξ. This shows that

$$\mathrm{d}\xi_p(X(p)) = 0 \ . \tag{A.2}$$

and thus that

$$\mathrm{d}\xi_p(Y^*(p)) = Y(\xi(p)) \ . \tag{A.3}$$

Y^* is actually the only horizontal vector field (*i.e.* vector field such that $Y^*(p) \in H_p$) on $\mathcal{U}$ that satisfies (A.3). This equation also shows that for $f \in \mathcal{F}(V)$,

$$Y^*(f^*) = Y(f)^* \tag{A.4}$$

Let us show that if $Y, Z \in \Gamma V$, we have

$$\pi([Y^*, Z^*]) = [Y, Z]^* \tag{A.5}$$

For $p \in D_t$, denote $L_p = \pi_p|_{T_pD_t}$. As T_pD_t and H_p are supplementary of $\mathcal{N}_p$, L_p is actually an isomorphism $T_pD_t \longrightarrow H_p$ (because the dimensions are equal and because $\ker L_p = T_pD_t \cap \mathcal{N}_p = \{0\}$). In order to show the equation above, we only need to show

$$L_p^{-1}(\pi([Y^*, Z^*])) = L_p^{-1}([Y, Z]^*) \ .$$

We know that for $v \in T_pD_t$, $v \in \pi_p(v) + \mathbb{R}X(p)$, *i.e.* $v \in L_p(v) + \mathbb{R}X(p)$. As L_p is surjective, this shows that for every $h \in H_p$, $L_p^{-1}(h) \in H + \mathbb{R}X(p)$, and as by definition, for every $c \in T_p\mathcal{H}$, $c \in \pi_p(c) + \mathbb{R}X(p)$, we get $L_p^{-1}(\pi_p(c)) \in c + \mathbb{R}X(p)$. With $c = [Y^*, Z^*]_p$, we obtain a $\lambda \in \mathbb{R}$ such that

$$L_p^{-1}(\pi([Y^*, Z^*]_p)) = [Y^*, Z^*]_p + \lambda X(p) \in T_pD_t \ .$$

On the other hand, by definition of $[Y, Z]^*$,

$$L_p^{-1}([Y, Z]^*) = (\mathrm{d}(\xi_t)_p)^{-1}\, Y(\xi(p)) \in T_pD_t \ .$$

To show the equality, let $f \in \mathcal{F}(D_t)$. As $\xi_t : D_t \longrightarrow V$ is a diffeomorphism, we can assume that f is of the type $f = r^*|_{D_t}$, where $r \in \mathcal{F}(V)$. However, by (A.2),

$$X(r^*) = X(r \circ \xi) = \mathrm{d}\xi(X)r = 0 \ .$$

We can now compute, thanks to (A.4),

$$\begin{aligned}
L_p^{-1}(\pi([Y^*, Z^*]_p))f &= ([Y^*, Z^*]_p + \lambda X(p))r^* \\
&= [Y^*, Z^*]_p r^* \\
&= Y^*(Z^*(r^*))_p - Z^*(Y^*(r^*))_p \\
&= Y(Z(r))_p^* - Z(Y(r))_p^* \\
&= ([Y, Z]r)_p^* \\
&= [Y, Z]_p^* r^* \\
&= L_p^{-1}([Y, Z]^*)r^* = L_p^{-1}([Y, Z]^*)f
\end{aligned}$$

As $L_p^{-1}([Y, Z]^*) \in [Y, Z]^* + \mathbb{R}X(p)$. This is true for every $f \in \mathcal{F}(D_t)$, which shows the equality wanted.
We will now introduce a "quotient" metric on V. Define the metric q on V by, for $Z, Y \in \Gamma V$, $p \in V$,

$$q(Z, Y)_p = g(Z^*, Y^*)_{\xi_0^{-1}(p)}$$

q is clearly a metric on V. Moreover, it is non-degenerate because $X^*, Y^* \in H$ and H is supplementary to the null direction of $\mathcal{H}$, so it contains no non-zero null vectors. This shows that q is non-degenerate, as the map $Y(p) \longmapsto Y^*(\xi_0^{-1}(p))$ is the map

$$\begin{array}{rccc}
L_{\xi_0^{-1}(p)} \circ \mathrm{d}(\xi_0)_p^{-1} : & T_pV & \longrightarrow & T_{\xi_0^{-1}(p)} \\
& v & \longmapsto & v^*
\end{array}$$

which is an isomorphism, as a composition of isomorphisms. Thus, (V, q) is a semi-Riemannian manifold. Denote by $\mathcal{D}$ its Levi-Civita connection. A way to link horizontal geodesics on $\mathcal{U}$ and geodesics on V will be to show a useful formula for $\mathcal{D}$. More precisely, we will show that for $Y, Z \in \Gamma V$, and for $p \in \mathcal{U}$,

$$(\mathcal{D}_Y Z)_{\xi(p)} = \mathrm{d}\xi_p \left(\pi \left(\nabla_{Y^*} Z^*(p)\right)\right) \tag{A.6}$$

This will comme from the fact that for every $p \in \mathcal{U}$,

$$q(Z, Y)_{\xi(p)} = h(Z^*, Y^*)_p \tag{A.7}$$

A priori, by definition of q, this only true for $p \in D$. To show (A.7), we will use the fact that on $\mathcal{U}$,

$$\mathcal{L}_X g = 0 \tag{A.8}$$

Indeed, as $\mathcal{L}$ is a tensor derivation, for $V, W \in \Gamma\mathcal{U}$,

$$\begin{aligned} Xg(V,W) &= \mathcal{L}_X g(V,W) + g(\mathcal{L}_X V, W) + g(V, \mathcal{L}_X W) \\ &= \mathcal{L}_X g(V,W) + g([X,V],W) + g(V,[X,W]) \end{aligned}$$

However, we know that $Xg(V,W) = g(\nabla_X V, W) + g(V, \nabla_X W)$, which shows that

$$(\mathcal{L}_X g)(V,W) = g(\nabla_V X, W) + g(\nabla_W X, V) = 0$$

because ∇X is still null and tangent to $\mathcal{H}$ (see Lemma 4.1), thus orthogonal to $\mathcal{H}$.
We can now show that for $p \in D$, for $v, w \in T_p D$, the function

$$t \longmapsto g(\mathrm{d}\varphi_t(v), \mathrm{d}\varphi_t(w)) \quad \text{is a constant.} \tag{A.9}$$

Indeed, we have the classical formula for the Lie derivative (see for example [ONe83, p. 250]) :

$$0 = \mathcal{L}_X g = \lim_{t\to 0} \frac{1}{t}\left(\varphi_t^* h - h\right)$$

thus, as $\varphi_{t+s} = \varphi_t \circ \varphi_s$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} g(\mathrm{d}\varphi_s(v), \mathrm{d}\varphi_t(w)) &= \lim_{t\to 0} \frac{1}{t}\left(g(\mathrm{d}\varphi_t(\mathrm{d}\varphi_s(v)), \mathrm{d}\varphi_t(\mathrm{d}\varphi_s(w)) - g(\mathrm{d}\varphi_s(v), \mathrm{d}\varphi_s(w))\right) \\ &= (\mathcal{L}_X g)(\mathrm{d}\varphi_s(v), \mathrm{d}\varphi_s(w)) = 0 \end{aligned}$$

and the function is indeed a constant . Moreover, as X is null and orthogonal to $T_p\mathcal{H}$, we have by (A.1), for $p \in D_t$,

$$g(Z^*, Y^*)_p = g((\mathrm{d}(\xi_t)_p)^{-1} Z(\xi(p), (\mathrm{d}(\xi_t)_p)^{-1} Y(\xi(p)) \ .$$

Thus, to show (A.7), we only need to show that for $x \in V$, the function

$$t \longmapsto g(\mathrm{d}(\xi_t^{-1})_x Z(x), \mathrm{d}(\xi_t^{-1})_x Y(x))$$

is a constant. Thanks to (A.9), and again with the fact that X is null and orthogonal to $T\mathcal{H}$, we only need to show that for every t,

$$\mathrm{d}(\xi_t^{-1})_x Z(x) \in \mathrm{d}\varphi_t(\mathrm{d}(\xi_0^{-1})_x Z(x)) + \mathbb{R}X \ .$$

We can compute, as clearly $\xi \circ \varphi_t = \xi$,

$$\mathrm{d}\xi_{\xi_t^{-1}(x)} \mathrm{d}(\xi_t^{-1})_x Z(x) = Z(x)$$

$$\mathrm{d}\xi_{\xi_t^{-1}(x)} \mathrm{d}\varphi_t(\mathrm{d}(\xi_0^{-1})_x Z(x)) = \mathrm{d}(\xi \circ \varphi_t)_{\xi_0^{-1}(x)}(\mathrm{d}(\xi_0^{-1})_x Z(x)) = \mathrm{d}\xi_{\xi_0^{-1}(x)}(\mathrm{d}(\xi_0^{-1})_x Z(x)) = Z(x) \ .$$

Which shows that $d\xi_{\xi_t^{-1}(x)} d(\xi_t^{-1})_x Z(x) = d\xi_{\xi_t^{-1}(x)} d\varphi_t(d(\xi_0^{-1})_x Z(x))$. Moreover, we know that $d(\xi_t^{-1})_x Z(x) \in TD_t$, thus if $R = L_p^{-1}(\pi(d\varphi_t(d(\xi_0^{-1})_x Z(x)))$, we know that

$$R \in d\varphi_t(d(\xi_0^{-1})_x Z(x)) + \mathbb{R}X$$

so $d\xi(R) = d\xi(d\varphi_t(d(\xi_0^{-1})_x Z(x))) = d\xi(d(\xi_t^{-1})_x Z(x))$. But $R \in TD_t$ by definition, and $d\xi_t$ is injective on TD_t because ξ_t is a diffeomorphism. Thus,

$$d(\xi_t^{-1})_x Z(x) = R \in d\varphi_t(d(\xi_0^{-1})_x Z(x)) + \mathbb{R}X$$

and we have (A.7), by the previous arguments. We can now show (A.6). We now know that

$$g(Z^*, Y^*) = q(Z, Y)^* \ .$$

We can compute, for every $W \in \Gamma V$, by the Koszul formula (see [ONe83, p. 61]) used twice, and by (A.5),

$$\begin{aligned}
g(\pi(\nabla_{Y^*} Z^*), W^*) &= g(\nabla_{Y^*} Z^*, W^*) \\
&= \frac{1}{2}(Y^* g(W^*, Z^*) + Z^* g(Y^*, W^*) - W^* g(Y^*, Z^*) \\
&\quad - g([Z^*, W^*], Y^*) - g([Y^*, W^*], Z^*) - g([Z^*, Y^*], W^*)) \\
&= \frac{1}{2}(Y q(W, Z) + Z q(Y, W) - W q(Y, Z) \\
&\quad - q([Z, W], Y) - q([Y, W], Z) - q([Z, Y], W))^* \\
&= q(\mathcal{D}_Y Z, W)^*
\end{aligned}$$

Thus, by (A.7), we have, by definition of $Y \longmapsto Y^*$,

$$\begin{aligned}
q(d\xi_p (\pi (\nabla_{Y^*} Z^*(p))), W)_{\xi(p)} &= g(d\xi_p (\pi (\nabla_{Y^*} Z^*(p)))^*, W^*)_p \\
&= g(\pi (\nabla_{Y^*} Z^*(p))^*, W^*)_p \\
&= q(\mathcal{D}_Y Z, W)_{\xi(p)} \ .
\end{aligned}$$

As this is true for every $W \in \Gamma V$, and as q is non-degenerate, we get (A.6). This equation is the fundamental result that will allow us to conclude.

Proof of Proposition 2.5

We start by proving the local existence of horizontal geodesics :

Proposition A.1

For every $p \in \mathcal{H}$ and $v \in H_p$, for $\varepsilon > 0$ small enough, there is a unique horizontal geodesic $\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathcal{H}$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

We will use compactness of $\mathcal{H}$ to deduce Proposition 2.5. We stay in the context of the preceding construction. The two following propositions make use of Equation (A.6) and will be used in the proof of Proposition A.1.

Proposition A.2

Let α be a curve on V and $p \in \mathcal{U}$ such that $\xi(p) = \alpha(0)$. There is a unique horizontal curve γ on $\mathcal{U}$ such that $\alpha = \xi \circ \gamma$.

Proof. We must chose γ of the form :

$$\gamma(s) = \xi_{f(s)}^{-1}(\alpha(s)) = \varphi(\xi_0^{-1}(\alpha(s)), f(s))$$

for some function f with $f(0)$ fixed such that $\gamma(0) = p$, $i.e.$ such that $p \in D_{f(0)}$. We can compute :

$$\begin{aligned}\gamma'(s) &= f'(s)X(\gamma(s)) + \mathrm{d}\varphi_{f(s)}\mathrm{d}(\xi_0^{-1})_{\alpha(s)}\alpha'(s) \\ &= f'(s)X(\gamma(s)) + \mathrm{d}(\varphi_{f(s)} \circ \xi_0^{-1})_{\alpha(s)}\alpha'(s) \\ &= f'(s)X(\gamma(s)) + \mathrm{d}(\xi_{f(s)}^{-1})_{\alpha(s)}\alpha'(s)\end{aligned}$$

And we have $\mathrm{d}(\xi_{f(s)}^{-1})_{\alpha(s)}\alpha'(s) \in T_{\gamma(s)}B_{f(s)}$. Thus, in order to have $\gamma'(s) \in H_{\gamma(s)}$, we have to chose f such that

$$f'(s) = -\mathrm{coord}_X(\mathrm{d}(\xi_{f(s)}^{-1})_{\alpha(s)}\alpha'(s))$$

where the term on the right is the coordinate on X of $\mathrm{d}(\xi_{f(s)}^{-1})_{\alpha(s)}\alpha'(s)$ in the decomposition $TC = H \oplus \mathcal{N}$. This term is a smooth function of $f(s)$, so this equation really is a first order ODE on f. Thus, we know that there is a unique solution f of this ODE such that $f(0)$ is fixed with p. Then γ defined as above is the unique horizontal curve with $\gamma(0) = p$ such that $\xi \circ \gamma = \alpha$. □

Proposition A.3

Let γ be a horizontal curve on $\mathcal{U}$, and $\alpha := \xi \circ \gamma$. Then,

$$\gamma \textit{ is a horizontal geodesic} \iff \alpha \textit{ is a geodesic of } V$$

Proof. For every s, $\alpha'(s) = \mathrm{d}\xi_{\gamma(s)}\gamma'(s)$. Extend α' to a vector field Z on a neighbourhood of α, $i.e.$ such that $Z(\alpha(s)) = \alpha'(s)$. Then we know that

$$\mathcal{D}_{\alpha'}\alpha'(s) = \mathcal{D}_Z Z(\alpha(s)) \ .$$

Moreover, if $\gamma(s) \in D_t$,

$$\begin{aligned}Z^*(\gamma(s)) &= \pi((\mathrm{d}\xi_t)^{-1}Z(\xi(\gamma(s)))) \\ &= \pi((\mathrm{d}\xi_t)^{-1}\alpha'(s)) \\ &= \pi(\gamma'(s)) \\ &= \gamma'(s)\end{aligned}$$

thus Z^* is a vector field on $\mathcal{U}$ that extends γ'. Equation (A.6) states that

$$\mathcal{D}_Z Z(\alpha(s)) = \mathrm{d}\xi_{\gamma(s)}\pi(\nabla_{Z^*}Z^*(\gamma(s)))$$

so we get

$$\mathcal{D}_{\alpha'}\alpha'(s) = \mathrm{d}\xi_{\gamma(s)}\pi(\nabla_{\gamma'}\gamma'(s)) \tag{A.10}$$

Now, if γ is a horizontal geodesic, $\pi(\nabla_{\gamma'}\gamma'(s)) = 0$ so by (A.10), α is a geodesic. If α is a geodesic, $\mathrm{d}\xi_{\gamma(s)}\pi(\nabla_{\gamma'}\gamma'(s)) = 0$. But we saw earlier that $L_{\gamma(s)}^{-1}(\pi(\nabla_{\gamma'}\gamma'(s))) \in \pi(\nabla_{\gamma'}\gamma'(s)) + \mathbb{R}X$ and $X \in \ker \mathrm{d}\xi$, so

$$L_{\gamma(s)}^{-1}(\pi(\nabla_{\gamma'}\gamma'(s))) \in T_{\gamma(s)}D_t \cap \ker \mathrm{d}\xi = \ker \mathrm{d}\xi_t = \{0\}$$

thus $\pi(\nabla_{\gamma'}\gamma'(s)) = 0$ and γ is a horizontal geodesic. □

We can now demonstrate the main results of this section.

Proof of Proposition A.1. As we need to show the result for $\varepsilon > 0$ small enough, it suffices to show the result in the open set $\mathcal{U} \supset p$.
Uniqueness : If γ_1 and γ_2 are two horizontal geodesics satisfying Proposition A.1, then by Proposition A.3, $\alpha_1 = \xi \circ \gamma_1$ and $\alpha_2 = \xi \circ \gamma_2$ are two geodesics of V starting in $\xi(p)$, with initial velocity $\mathrm{d}\xi(v)$, so are equal by uniqueness of geodesics. The uniqueness of Proposition A.2 then shows us that $\gamma_1 = \gamma_2$.
Existence : Let α be a geodesic on V such that $\alpha(0) = \xi(p)$ and $\alpha'(0) = \mathrm{d}\xi(v)$. By Proposition A.2, there is a horizontal curve γ on $\mathcal{U}$ such that $\xi \circ \gamma = \alpha$, and such that $\gamma(0) = p$. Proposition A.3 shows that γ is a horizontal geodesic. Finally, we have $\mathrm{d}\xi(\gamma'(0)) = \mathrm{d}\xi(v)$, with $v, \gamma'(0) \in H_p$. However, we saw that

$$\mathrm{d}\xi_p|_{H_p} = \mathrm{d}(\xi_t)_p \circ L_p^{-1}$$

is injective, so $\gamma'(0)$ must be equal to v, hence the existence □

Proof of Proposition 2.5. Proposition A.1 shows us that there is $a < b \in \mathbb{R}$, possibly infinite, and an inextensible horizontal geodesic $\gamma : (a, b) \longrightarrow \mathbb{R}$ such that $\gamma(0) = p$, $\gamma'(0) = v$. Suppose for example that $b < \infty$. Let $b_n \longrightarrow b$ be a converging sequence of (a, b). Recall that as H is transversal to the null direction of $\mathcal{H}$, H is actually spacelike (H cannot contain any non-zero null vector nor any timelike vector because then it would contain two independent null vectors, see for example [ONe83, p. 141]). The computation

$$g(\gamma', \gamma')' = 2g(\nabla_{\gamma'}\gamma', \gamma') = 2g(\pi(\nabla_{\gamma'}\gamma'), \gamma') = 0$$

shows that the spacelike norm of γ is constant. Thus, by compactness of $\mathcal{H}$, and by Proposition A.3, γ' lies in a compact set of $T\mathcal{H}$, and thus we can chose b_n such that $\gamma(b_n)$ and $\gamma'(b_n)$ converge. But then, Proposition A.1 allows us to extend γ with a horizontal geodesic starting from $\lim_n \gamma(b_n)$ with initial velocity $\lim_n \gamma'(b_n)$, which is a contradiction by inextendibility of γ. Thus $b = \infty$, and similarly, $a = -\infty$. □

B Mathematical relativity

B.1 The notion of spacetime

Definition B.1

A Lorentzian manifold is a pair $(\mathcal{M}, g)$ where $\mathcal{M}$ is a smooth manifold, and g is a $\mathscr{C}^\infty$ non-degenerate metric on $\mathcal{M}$ with signature $(-, +, \dots, +)$ at every point of $\mathcal{M}$.

If $X \in T\mathcal{M}$,we define the *causal character* of X as

$$\begin{aligned} &\textit{timelike} \text{ if } g(X,X) < 0 \\ &\textit{spacelike} \text{ if } g(X,X) > 0 \\ &\textit{lightlike} \text{ if } g(X,X) = 0 \text{ and } X \neq 0\,. \end{aligned}$$

We also say that X is *causal* if X is timelike or lightlike, and that X is *null* if X is lightlike or zero. We extend pointwisely the previous definitions to vector fields $X \in \Gamma\mathcal{C}$: X is said to be *timelike*/*spacelike*/*lightlike*/*causal*/*null* if $X(p)$ is timelike/spacelike/lightlike/causal/null at every point $p \in \mathcal{M}$. We also extend these definitions to curves $\gamma : I \longrightarrow \mathcal{M}$.
The *causal cone* of $p \in \mathcal{M}$ is the set of causal vectors of $T_p\mathcal{M}$. The causal cone of p has two connected components.
A Lorentzian manifold $(\mathcal{M}, g)$ is said to be *time-orientable* if there is a smooth choice of a connected component of the causal cone. More precisely, $(\mathcal{M}, g)$ is time-orientable if there is a smooth global timelike vector field $V \in \Gamma\mathcal{M}$.
At every point $p \in \mathcal{M}$, the connected component of the causal cone containing $V(p)$ is called the *future causal cone* of p, and the other connected component is the *past causal cone*. A causal vector $X \in T\mathcal{M}$ is *future* if it is in the future causal cone, and *past* otherwise.

Definition B.2

A spacetime is a time-oriented non-compact connected Lorentzian manifold $(\mathcal{M}, g)$.

The non-compact hypothesis is here because compact Lorentzian manifolds are pathologically non-physical. For example, they always contain closed timelike curves, which is not a very physical property (time travel, violation of causality...).

The most simple example of spacetime is the Minkowski space $\mathbb{R}^{n+1}$, endoced with the Lorentzian metric $g = -\mathrm{d}t^2 + (\mathrm{d}x^1)^2 + ... + (\mathrm{d}x^n)^2$. Notice that for every spacetime $(\mathcal{M}, g)$ and $p \in \mathcal{M}$, the spacetime $(T_p\mathcal{M}, g_p)$ is isometric to the Minkowski space of dimension $n + 1 = \dim \mathcal{M}$.

Proposition B.1

The causal character of geodesics on $\mathcal{M}$ *is constant.*

B.2 Causality theory

The main definitions and results of causality theory that we will use are presented here. Let $(\mathcal{M}, g)$ be a spacetime and $S \subseteq \mathcal{M}$. We define :

(i) the *chronological future* of S as :

$$I^+(S) = \left\{ p \in \mathcal{M} \,\middle|\, \text{there is a future-directed timelike curve leaving } S \text{ ending at } p \right\}$$

(ii) the *causal future* of S as :

$$J^+(S) = \left\{ p \in \mathcal{M} \,\middle|\, \text{there is a future-directed causal curve leaving } S \text{ ending at } p \right\}$$

(iii) the *chronological past* of S as :

$$I^-(S) = \left\{ p \in \mathcal{M} \,\middle|\, \text{there is a past-directed timelike curve leaving } S \text{ ending at } p \right\}$$

(iv) the *causal past* of S as :

$$J^- S) = \left\{ p \in \mathcal{M} \,\middle|\, \text{there is a future-directed causal curve leaving } S \text{ ending at } p \right\} .$$

If $F \subseteq \mathcal{M}$ and $S \subseteq \mathcal{M}$, we define

$$I^+(S, F) = \left\{ p \in \mathcal{M} \,\middle|\, \text{there is a future-directed timelike curve contained in } F \text{ leaving } S \text{ ending at } p \right\}$$

and similarly for the past and causal future and past versions.

Proposition B.2

The causality relation I *is open.*

This result means that if $p \in I^+(q)$, the the same is true for (p', q') in an open neighbourhood of (p, q). A set $A \subseteq \mathcal{M}$ is said to be *achronal* if $I^+(A) \cap A = \varnothing$, that is, no timelike curve can connect two points of A.

The *edge* of an achronal set A is defined as the set of points $p \in \overline{A}$ such that every neighbourhood $\mathcal{U}$ of p contains a timelike curve from $I^-(p, \mathcal{U})$ to $I^+(p, \mathcal{U})$ that doesn't intersect A.

Definition B.3

A partial Cauchy surface is an acausal and edgeless hypersurface of $\mathcal{M}$.

The following results are respectively [ONe83, Prop 14.25], [ONe83, Cor 14.27] and [ONe83, Prop 14.53 (1)], and can be used to show that Cauchy horizons of connected partial Cauchy surfaces are $\mathscr{C}^0$ hypersurfaces, as shown in Section 2.2.

Proposition B.3

An achronal set $A \subseteq \mathcal{M}$ is a topological hypersurface of $\mathcal{M}$ if and only if A and its edge are disjoint.

A set $P \subseteq \mathcal{M}$ is said to be *past* if $I^-(P) \subset P$.

Corollary B.1

The boundary of a future set is a closed achronal topological hypersurface.

Proposition B.4

Let $S \subseteq \mathcal{M}$ be a closed acausal topological hypersurface. Then

$$\mathcal{H}^+(S) = I^+(S) \cap \partial D^+(S) \ .$$

C Induction proof of the smoothness of the affine length

In the context of Section 5 (non-vacuum case), this section proves by induction the smoothness of the affine length Λ. We begin with two lemmas that clarify the structure of $\nabla^k_{X_1^\rho \dots X_k^\rho} X_r^\rho$ compared to $(\nabla^k_{X_1 \dots X_k} X_r)^\rho$, where the $X_i's$ are vector fields, and then Proposition C.1 easily concludes.

If T is a covariant tensor field and the X_i's are vector fields, by the expression

$$T\left(\nabla^{\leq k}_{X_{i_1} \dots X_{i_m}} X_r\right)$$

we mean the evaluation of T on terms of the type $\nabla^m_{X_{i_1} \dots X_{i_m}} X_r$ where $m \leq k$.

Lemma C.1

Let $(X_i)_{i \in \mathbb{N}}$ be local vector fields around $p \in \mathcal{H}$. For every $k \in \mathbb{N}$, there is a tensor field $\eta^{(k)}$ such that for $\rho \geq 0$,

$$\nabla^k_{X^\rho_{i_1} \dots X^\rho_{i_k}} X_r^\rho = \left(\nabla^k_{X_{i_1} \dots X_{i_k}} X_r\right)^\rho + \left[\int_0^\rho \eta^{(k)} \left(\nabla^{\leq k-1}_{X^s_{j_1} \dots X^s_{j_m}} X^s_r\right) \mathrm{d}s\right] n \ .$$

Proof. We proceed by induction on k. Denote $Y_k = \nabla^k_{X_{i_1} \dots X_{i_k}} X_r$. The case $k = 0$ is clear, and the case $k = 1$ is the consequence of the following fact that was proved in the demonstration of Proposition 5.14 :

$$\nabla_{X^\rho} Y^\rho = (\nabla_X Y)^\rho + \left[\int_0^\rho \eta(X^s, Y^s) \mathrm{d}s\right] n \ .$$

Suppose that the result holds for $k \in \mathbb{N}$, and let us prove that it is still true for $k+1$. We have, by Proposition 5.8 and the induction hypothesis,

$$\begin{aligned}
\mathcal{L}_n[\nabla^{k+1}_{X^\rho, X^\rho_{i_1} \dots X^\rho_{i_k}} X^\rho_r] &= (\mathcal{L}_n \nabla)(X^\rho) \nabla^k_{X^\rho_{i_1} \dots X^\rho_{i_k}} X^\rho_r + \nabla_{X^\rho}\left[\mathcal{L}_n\left[\nabla^k_{X^\rho_{i_1} \dots X^\rho_{i_k}} X^\rho_r\right]\right] \\
&= \eta(X^\rho, \nabla^k_{X^\rho_{i_1} \dots X^\rho_{i_k}} X^\rho_r) n + \nabla_{X^\rho}\left[\eta^{(k)}(\nabla^{\leq k-1}_{X^\rho_{j_1} \dots X_{j_m} \rho} X^\rho_r) \cdot n\right] \\
&= \Big(\eta(X^\rho, \nabla^k_{X^\rho_{i_1} \dots X^\rho_{i_k}} X^\rho_r) + \omega(X^\rho) \eta^{(k)}(\nabla^{\leq k-1}_{X^\rho_{j_1} \dots X^\rho_{j_m}} X^\rho_r) \\
&\quad + \nabla_{X^\rho} \eta^{(k)}(\nabla^{\leq k-1}_{X^\rho_{j_1} \dots X^\rho_{j_m}} X^\rho_r) + \eta^{(k)}(\nabla^{\leq k}_{X^\rho_{j_1} \dots X^\rho_{j_m}} X^\rho_r)\Big) n \\
&:= \eta^{(k+1)}\left(\nabla^{\leq k}_{X^\rho_{j_1} \dots X^\rho_{j_m}} X^\rho_r\right) n \ .
\end{aligned}$$

Which shows, by uniqueness of the solution of the ODE $\mathcal{L}_n Z = 0$, $Z(0) = 0$, that

$$\nabla^{k+1}_{X^\rho, X^\rho_{i_1} \dots X^\rho_{i_k}} X^\rho_r = (\nabla^{k+1}_{X_{i_1} \dots X_{i_k}} X_r)^\rho + \left[\int_0^\rho \eta^{(k+1)} \left(\nabla^{\leq k}_{X^s_{j_1} \dots X^s_{j_m}} X^s_r \right) \mathrm{d}s \right] n$$

hence the induction. ☐

Lemma C.2

Let $(X_i)_{i\in\mathbb{N}}$ be local vector fields around $p \in \mathcal{H}$. For every $k \in \mathbb{N}$, there is a vector field Y_k and a function λ_k such that for $\rho \geq 0$,

$$\nabla^k_{X^\rho_{i_1} \dots X^\rho_{i_k}} X^\rho_r = (Y_k)^\rho + \lambda_k(\rho) n$$

and such that λ_k is dominated by a polynomial function in ρ with continuous coefficients.

Proof. We proceed again by induction. The case $k = 0$ is clear, and the case $k = 1$ is the consequence the demonstration of Proposition 5.14 and of the fact that we saw in the demonstration of Proposition 5.10 that $X^\rho = X^\rho_H + \lambda(\rho) n$ where $X^\rho_H \in H$ and where λ is bounded by an affine function (these facts show that $\int_0^\rho \eta(X^s, Y^s)\mathrm{d}s$ is bounded by a polynomial in ρ).

Suppose that the result holds for $k \in \mathbb{N}$, and let us prove that it is still true for $k+1$. With Lemma C.1, we only need to check that the function

$$\int_0^\rho \eta^{(k+1)} \left(\nabla^{\leq k}_{X^s_{j_1} \dots X^s_{j_m}} X^s_r \right) \mathrm{d}s$$

is dominated by a polynomial in ρ. This is true because applying the induction hypothesis, we have that $\nabla^{\leq k-1}_{X^s_{j_1} \dots X^s_{j_m}} X^s_r = (Y_k)^\rho + \lambda_k n$ with λ_k dominated by a polynomial. We then have by decomposing $(Y_k)^\rho = (Y_k)^\rho_H + \eta n$ that, for $r \leq k-1$,

$$\nabla^r_{X^s_{j_1} \dots X^s_{j_m}} X^s = (Y_k)^\rho_H + \lambda'_k n$$

where λ'_k is dominated by a polynomial. Thus,

$$\int_0^\rho \eta^{(k+1)} \left(\nabla^{\leq k}_{X^s_{j_1} \dots X^s_{j_m}} X^s_r \right) \mathrm{d}s$$

is just a sum of integrals of polynomial-dominated functions times tensor fields (contracted with n) evaluated on horizontal vectors, and we conclude with the compactness of the unitary subbundle of H that it is polynomial-dominated. ☐

Proposition C.1

For every $k \in \mathbb{N}$, Λ is $\mathscr{C}^k$ and there is $\ell \in \mathbb{N}$ and tensor fields U_i, V_i on $\mathcal{H}$ such that

$$\nabla^k \Lambda(X_1, ..., X_k) = \int_0^\infty \left[\sum_{i=1}^{\ell} U_i(\nabla^{\leq k}_{X_{i_1} \dots X_{i_m}} X_r) V_i(\nabla^{\leq k}_{X^\rho_{i_1} \dots X^\rho_{i_m}} X^\rho_r) \right] \exp\left(\int_{\varphi_p}^\rho \kappa \right) \mathrm{d}\rho\,.$$

Proof. We proceed by induction. The result is clear for $k = 0$ and $k = 1$ with Proposition 5.10. Suppose that the result holds for k and let $X \in T\mathcal{H}$. Denote

$$F_\rho = \left[\sum_{i=1}^{\ell} U_i(\nabla^{\leq k}_{X_{i_1} \dots X_{i_m}} X_r) V_i(\nabla^{\leq k}_{X^\rho_{i_1} \dots X^\rho_{i_m}} X^\rho_r) \right] \exp\left(\int_{\varphi_p}^\rho \kappa \right) \mathrm{d}\rho\,.$$

For $\rho \geq 0$, we have

$$\nabla_X F_\rho = (\omega(X^\rho) - \omega(X))F_\rho + \Big[\sum_{i=1}^{\ell}(\nabla_X U_i(\nabla^{\leq k}_{X_{i_1}\dots X_{i_m}} X_r) + U_i(\nabla^{\leq k+1}_{X_{i_1}\dots X_{i_m}} X_r))V_i(\nabla^{\leq k}_{X^\rho_{i_1}\dots X^\rho_{i_m}} X^\rho_r)$$
$$+ U_i(\nabla^{\leq k}_{X_{i_1}\dots X_{i_m}} X_r)(\nabla_{X^\rho} V_i(\nabla^{\leq k}_{X^\rho_{i_1}\dots X^\rho_{i_m}} X^\rho_r) + V_i(\nabla^{\leq k+1}_{X^\rho_{i_1}\dots X^\rho_{i_m}} X^\rho_r))\Big] \exp\left(\int_{\varphi_p}^{\rho} \kappa\right) \mathrm{d}\rho$$
$$= \left[\sum_{i=1}^{\ell'} U_i'(\nabla^{\leq k+1}_{X_{i_1}\dots X_{i_m}} X_r) V_i'(\nabla^{\leq k+1}_{X^\rho_{i_1}\dots X^\rho_{i_m}} X^\rho_r)\right] \exp\left(\int_{\varphi_p}^{\rho} \kappa\right) \mathrm{d}\rho\ .$$

Notice that using the splitting $T\mathcal{H} = H \oplus \mathbb{R}n$, Lemma C.2 shows that each term of the form $\nabla^{\leq k+1}_{X_{i_1}\dots X_{i_m}} X_r$ is the sum of a horizontal vector with a polynomial-dominated function times n. Thus,

$$\sum_{i=1}^{\ell'} U_i'(\nabla^{\leq k+1}_{X_{i_1}\dots X_{i_m}} X_r) V_i'(\nabla^{\leq k+1}_{X^\rho_{i_1}\dots X^\rho_{i_m}} X^\rho_r)$$

is polynomial-dominated, as a sum of polynomial-dominated functions times tensor fields (contracted with n) evaluated on horizontal vectors. As there is $K > 0$ such that $\kappa < -K$, we conclude by the theorem of differentiation under the integral sign that Λ is $\mathscr{C}^{k+1}$ and that $\nabla^{k+1}\Lambda$ is given by

$$\nabla^{k+1}\Lambda(X, X_1, \dots, X_n) = \int_0^\infty (\nabla_X F_\rho)\, \mathrm{d}\rho - \sum_{i=1}^{k} \nabla^k \Lambda(X_1, \dots, \nabla_X X_i, \dots, X_k)$$

which is of the announced form, by the induction hypothesis. □

A direct corollary of Proposition C.1 is :

Corollary C.1

The function Λ is smooth.

D Behavior of general null generators

Possible behaviours of complete generators

Let $\mathcal{H}$ be a connected compact Cauchy horizon in a spacetime where the dominant energy condition holds. Section 5.2 shows that if $\mathcal{H}$ is non-degenerate, its null generators are future incomplete. Denote $\Gamma^+_{\mathcal{N}}\mathcal{H}$ the set of smooth future-directed nowhere-zero null vector fields on $\mathcal{H}$. We actually showed the following result :

Theorem D.1

Let $\mathcal{H}$ be a compact Cauchy horizon. The following are equivalent :

(i) every null generator of $\mathcal{H}$ is future-incomplete

(ii) $\mathcal{H}$ contains a future-incomplete null generator

(iii) for every $p \in \mathcal{H}$ and $Z \in \Gamma^+_{\mathcal{N}}\mathcal{H}$, $\int_{\varphi^Z_p} \omega_Z = -\infty$

(iv) there is $p \in \mathcal{H}$ and $Z \in \Gamma^+_{\mathcal{N}}\mathcal{H}$ such that $\int_{\varphi^Z_p} \omega_Z = -\infty$

(v) there is $X \in \Gamma^+_{\mathcal{N}}\mathcal{H}$ such that $\omega_X(X) < 0$.

*A horizon that satisfies this properties will be called here an **incomplete horizon**.*

Proof. (i) $\implies$ (ii) and (iii) $\implies$ (iv) are clear. (ii) $\implies$ (iii) is the object of the ribbon argument and is proved by Lemma 5.4 and Proposition 5.7. (iv) $\implies$ (v) is the object of Lemma 5.5 and Propositions 5.8 and 5.9. Finally, (v) $\implies$ (i) is proved by the demonstration of Corollary 5.1. $\square$

We can wonder the possible behaviors of the generators of a non-incomplete horizon $\mathcal{H}$. Let $Z \in \Gamma^+_{\mathcal{N}}\mathcal{H}$. We know that for every $p \in \mathcal{H}$, $\int_{\varphi_p^Z} \omega_Z$ doesn't diverge to $-\infty$. This integral could instead diverge to $+\infty$. We will see that this is actually impossible, as a consequence of the past-incompleteness of the null generators of a compact future Cauchy horizon. This fact was first stated by Hawking and Ellis in [HE73, Lemma 8.5.5] and was then rigorously proved in [Min14]. We start by proving the following result :

Lemma D.1

Let $\mathcal{H}$ be a compact Cauchy horizon. The following are equivalent :

(i) for every $p \in \mathcal{H}$ and $Z \in \Gamma^+_{\mathcal{N}}\mathcal{H}$, $\displaystyle\int_{\varphi_p^Z} \omega_Z = +\infty$

(ii) there is $p \in \mathcal{H}$ and $Z \in \Gamma^+_{\mathcal{N}}\mathcal{H}$ such that $\displaystyle\int_{\varphi_p^Z} \omega_Z = +\infty$

(iii) there is $X \in \Gamma^+_{\mathcal{N}}\mathcal{H}$ such that $\omega_X(X) > 0$.

Proof. (i) $\implies$ (ii) is clear. For the converse, notice that the ribbon argument can be used like in the proof of Proposition 5.7, only replacing $-\infty$ with $+\infty$, to show that for $Z \in \Gamma^+_{\mathcal{N}}\mathcal{H}$, the set of points $p \in \mathcal{H}$ such that $\int_{\varphi_p^Z} \omega_Z = +\infty$ is open and closed in $\mathcal{H}$. Thus, if $Z \in \Gamma^+_{\mathcal{N}}\mathcal{H}$ and $p_0 \in \mathcal{H}$ are such that $\int_{\varphi_{p_0}^Z} \omega_Z = +\infty$ then for every $p \in \mathcal{H}$, $\int_{\varphi_p^Z} \omega_Z = +\infty$. Let $X \in \Gamma^+_{\mathcal{N}}\mathcal{H}$. There is a smooth strictly positive function $f : \mathcal{H} \longrightarrow \mathbb{R}$ such that $X = fZ$. Let $p \in \mathcal{H}$, and let $F : [0, \infty) \longrightarrow [0, \infty)$ be the reparametrization of φ_p^Z into φ_p^X, *i.e.* such that $\varphi_p^X = \varphi_p^Z \circ F$. We have

$$f(\varphi_p^Z(F(t))Z(\varphi_p^Z(F(t)) = X(\varphi_p^X(t)) = \frac{\mathrm{d}\varphi_p^X}{\mathrm{d}t}(t) = \frac{\mathrm{d}(\varphi_p^Z \circ F)}{\mathrm{d}t}(t) = F'(t)\frac{\mathrm{d}\varphi_p^Z}{\mathrm{d}t}(F(t)) = F'(t)Z(\varphi_p^Z(F(t)) \, .$$

Thus F is the solution of the ODE $F(0) = 0$, $F'(t) = f(\varphi_p^Z(F(t))$. Recall that by Proposition 5.6, $\omega_X = \omega_Z + \mathrm{d}(\log f)$. We can now compute, for $\rho \geq 0$,

$$\begin{aligned}
\int_{\varphi_p^X}^{\rho} \omega_X &= \int_0^{\rho} \omega_X(X(\varphi_p^X(t)))\mathrm{d}t \\
&= \int_0^{\rho} f(\varphi_p^Z(F(t)))\omega_Z(Z(\varphi_p^Z(F(t))))\mathrm{d}t + \left[\log f(\varphi_p^X(t))\right]_0^{\rho} \\
&= \int_0^{\rho} \omega_Z(Z(\varphi_p^Z(F(t))))F'(t)\mathrm{d}t + \log(f(\varphi_p^X(\rho))/f(p)) \\
&= \int_{\varphi_p^Z}^{F(\rho)} \omega_Z + \log(f(\varphi_p^X(\rho))/f(p)) \, .
\end{aligned}$$

Notice that, as $f > 0$ is continuous on the compact set $\mathcal{H}$, the quantity $\log(f(\varphi_p^X(\rho))/f(p))$ is bounded. Thus, by choice of Z, and as $F(\rho) \to +\infty$, we have $\int_{\varphi_p^X} \omega_X = +\infty$. Hence the implication (ii) $\implies$ (i). (iii) $\implies$ (ii) is clear because for such an X, $\omega_X(X) > K$ for $K > 0$ a constant. For the converse, notice that the exact same reasoning as in Lemma 5.5 and Propositions 5.8 and 5.9 works, only replacing $-\infty$ by $+\infty$ and < 0 by > 0. $\square$

Proposition D.1

Let $\mathcal{H}$ be a compact Cauchy horizon. For every $Z \in \Gamma^+_{\mathcal{N}}\mathcal{H}$ and $p \in \mathcal{H}$, $\displaystyle\int_{\varphi_p^Z} \omega_Z \neq +\infty$.

Proof. Suppose that the result is false. By Lemma D.1, there is $X \in \Gamma^+_{\mathcal{N}}\mathcal{H}$ such that $\omega_X(X) > 0$. By Proposition 5.5, the formula

$$L_X(p) = \int_0^\infty \exp\left(\int_{\varphi_p^X}^\rho \omega_X\right) \mathrm{d}\rho \geq \int_0^\infty 1\mathrm{d}\rho = +\infty$$

shows that the null generators of $\mathcal{H}$ are future-complete. Moreover, as proved in [Min14], the null generators of $\mathcal{H}$ are past-complete. Thus, let $p \in \mathcal{H}$ and let $\gamma : \mathbb{R} \longrightarrow \mathcal{H}$ be the null geodesic on $\mathcal{H}$ such that $\gamma(0) = p$ and $\gamma'(0) = X(p)$. By Proposition 2.1, there is a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}^*_+$ such that for every $t \in \mathbb{R}$,

$$\gamma'(t) = f(t)X(\gamma(t)) \ .$$

As γ is a geodesic,

$$\begin{aligned} 0 &= \nabla_{\gamma'}\gamma' \\ &= f'X \circ \gamma + f^2(\nabla_X X) \circ \gamma \\ &= (f' + f^2\omega_X(X) \circ \gamma)X \ . \end{aligned}$$

Thus, $f' = -f^2\omega_X(X) \circ \gamma$, *i.e.* $(1/f)' = \omega_X(X) \circ \gamma$. Integrating between 0 and $t \in \mathbb{R}$ gives, as $f(0) = 1$,

$$\frac{1}{f(t)} - 1 = \int_0^t \omega_X(X(\gamma(s)))\mathrm{d}s \quad i.e. \quad f(t) = \frac{1}{1 + \int_0^t \omega_X(X(\gamma(s)))\mathrm{d}s} \ .$$

But then, for $t > 0$,

$$f(-t) = \frac{1}{1 + \int_0^{-t} \omega_X(X(\gamma(s)))\mathrm{d}s} = \frac{1}{1 - \int_0^t \omega_X(X(\gamma(-s)))\mathrm{d}s} \ .$$

As there is a constante $K > 0$ such that $\omega_X(X) > K$, the term $1 - \int_0^t \omega_X(X(\gamma(-s)))\mathrm{d}s$ vanishes for a finite $t > 0$, which contradicts the fact that f is defined up to $-\infty$, *i.e.* the fact that γ is past-complete. This is a contradiction, hence the result. □

The result of Proposition D.1 shows that the only possibility for a compact Cauchy horizon $\mathcal{H}$ to be non-incomplete is that for every $Z \in \Gamma^+_{\mathcal{N}}\mathcal{H}$ and $p \in \mathcal{H}$, $\int_{\varphi_p^Z} \omega_Z \neq \pm\infty$. Hence the following definition :

Definition D.1

A compact Cauchy horizon $\mathcal{H}$ is said to be ***oscillating*** *if it is not incomplete.*

Theorem D.1 and Corollary D.1 show that a compact Cauchy horizon is oscillating if and only if for every $Z \in \Gamma^+_{\mathcal{N}}\mathcal{H}$ and $p \in \mathcal{H}$, $\int_{\varphi_p^Z} \omega_Z$ doesn't diverge to $\pm\infty$, if and only there is $Z \in \Gamma^+_{\mathcal{N}}\mathcal{H}$ and $p \in \mathcal{H}$ such that $\int_{\varphi_p^Z} \omega_Z$ doesn't diverge to $\pm\infty$. We now introduce a related but different kind of horizon :

Theorem D.2

Let $\mathcal{H}$ be a compact Cauchy horizon. The following are equivalent :

(i) for every $p \in \mathcal{H}$ and $Z \in \Gamma^+_{\mathcal{N}}\mathcal{H}$, $\rho \longmapsto \int_{\varphi_p^Z}^\rho \omega_Z$ is bounded

(ii) there is $p \in \mathcal{H}$ and $Z \in \Gamma^+_{\mathcal{N}}\mathcal{H}$ such that $\rho \longmapsto \int_{\varphi_p^Z}^\rho \omega_Z$ is bounded.

A horizon that satisfies this properties will be called here a ***bounded horizon****.*

Proof. (i) $\Longrightarrow$ (ii) is clear. For the converse, let p, Z be as in (ii). The ribbon argument used exactly like in the proof of Proposition 5.7 shows that the set of points $p \in \mathcal{H}$ such that $\rho \longmapsto \int_{\varphi_p^Z}^\rho \omega_Z$

is bounded is open and closed in $\mathcal{H}$. Thus, for every $p' \in \mathcal{H}$, $\rho \longmapsto \int_{\varphi^Z_{p'}}^{\rho} \omega_Z$ is bounded. For $f : \mathcal{H} \longrightarrow \mathbb{R}$ a strictly positive smooth function, and $X = fZ$, We previously proved that

$$\int_{\varphi^X_p}^{\rho} \omega_X = \int_{\varphi^Z_p}^{F(\rho)} \omega_Z + \log(f(\varphi^X_p(\rho))/f(p))$$

where F is the reparametrization of φ^Z_p into φ^X_p. As $\log(f(\varphi^X_p(\rho))/f(p))$ is bounded, this formula shows that for every $p' \in \mathcal{H}$, $\rho \longmapsto \int_{\varphi^Z_{p'}}^{\rho} \omega_Z$ is bounded. □

Notice that a bounded horizon is also an oscillating horizon. The converse is not true, as there could be a point p such that $\rho \longmapsto \int_{\varphi^Z_p}^{\rho} \omega_Z$ is not bounded and doesn't diverge to $-\infty$. However, the following facts hold :

(i) an oscillating horizon with a closed null generator is bounded
(ii) according to [Haw92], the set of compact Cauchy horizons with a closed null generator is *generic*.

More precisely, in his article *Chronology protection conjecture* [Haw92], Stephen Hawking states that given a spacetime $(\mathcal{M}, g)$ and a surface $\Sigma \subseteq \mathcal{M}$, the set of spacetime-metrics g' on $\mathcal{M}$ such that the g'-future Cauchy horizon $\mathcal{H}'$ of Σ has a closed null generator is dense in the set of spacetime-metrics on $\mathcal{M}$ (with a few supplementary hypothesis, it is even an open and dense subset).

Intuitively, this property is supported by the following result, that can be found in [Min14, Th 2.3] : even if $\mathcal{H}$ doesn't admit a closed null generator, because of compactness $\mathcal{H}$ will have an *almost closed* generator, *i.e.* that will future-accumulate on every point of itself. Arbitrary small perturbations of the metric will then be able to close this generator. Hawking used this fact with the objective of using closed null generators to show that, with physical assumptions, closed timelike curves cannot develop from a non-compact initial surface, which suggests that time travel is prevented by the laws of physics, except maybe at the quantum scale.

It worth to mention that Hawking only provides a sketch of proof and intuitive arguments, thus the genericity of closed generators has yet to be confirmed. We should also mention that Hawking worked in collaboration with Kip Thorne on the generecity of fountains, *i.e.* closed *and attractive* closed generators in Cauchy horizons, for example in [Tho93]. The genericity of fountains has since been disproved in [CI94], but the genericity of closed generators is still open.

Here, supposing that it holds, (ii) will be used to classify compact Cauchy horizons into two categories : those that contain a closed generators, and those that don't. This last category is *unstable*, *i.e.* arbitrary small perturbation of such a horizon will place it in the other category, and thus cannot occur if quantum fluctuations are taken into account. We will now prove (i).

Proposition D.2

Let $\mathcal{H}$ be a compact Cauchy horizon with a closed null generator. Then $\mathcal{H}$ is oscillating if and only if it is bounded.

Proof. Let $Z \in \Gamma^+_{\mathcal{N}}\mathcal{H}$ and let $p \in \mathcal{H}$ such that the null generator γ closes, where $\gamma(0) = p$, $\gamma'(0) = Z(p)$. Then the flow φ^Z_p is periodic. Indeed, as φ^Z_p is a reparametrization of γ, there is $T > 0$ such that $\varphi^Z(p, T) = p$. But then the uniqueness of the solution of the ODE $(\varphi^Z_p)' = Z \circ \varphi^Z_p$, $\varphi^Z_p(0) = p$ shows that $\varphi^Z_p(T + \cdot\,) = \varphi^Z_p$.

Thus, the function $\omega_Z(Z) \circ \varphi_p^Z$ is periodic, and for $\rho \geq 0$,

$$\int_{\varphi_p}^{\rho} \omega_Z = [\rho]_T \int_{\varphi^p}^{T} \omega_Z + \int_{[\rho]_T}^{\rho} \omega_Z(Z(\varphi_p^Z(s)))\mathrm{d}s$$

where where $[\rho]_T$ is the integer part of ρ/T. Notice that

$$\left| \int_{[\rho]_T}^{\rho} \omega_Z(Z(\varphi_p^Z(s)))\mathrm{d}s \right| \leq T \|\omega_Z(Z)\|_{L^\infty(\mathcal{H})}$$

is bounded, and that

$$[\rho]_T \int_{\varphi^p}^{T} \omega_Z$$

is bounded if and only if it doesn't diverge to $\pm\infty$ if and only if $\int_{\varphi^p}^{T} \omega_Z = 0$. This proves, as wished, that in that case, $\mathcal{H}$ is bounded if and only if it is oscillating. □

Proposition D.2 shows that an oscillating compact Cauchy horizon can be non-bounded, but in that case it is unstable, as shown by (ii). Abusing vocabulary a bit, let us call *ergodic* a compact Cauchy horizon that has no closed null generator. Sections 5.3 and D have shown the following result :

Theorem D.3

Let $\mathcal{H}$ be a compact Cauchy horizon in a spacetime satisfying the dominant energy condition. Then one of the following holds :
(i) $\mathcal{H}$ is ergodic and thus is unstable
(ii) $\mathcal{H}$ is incomplete and thus is symmetric
(iii) $\mathcal{H}$ is bounded.

As seen in Section 5.3, incomplete compact Cauchy horizons present some sort of symmetry, in the sense that they have an homogeneity vector field . In the vacuum case, they are even Killing horizons. Thus, if [Haw92] is correct, compact Cauchy horizons can be divided into the three categories : unstable, symmetric, and bounded. This suggests that it might be worth to focus on bounded horizons, to try to find nice properties that they could satisfy, for example the existence of a vanishing surface gravity, as explained in conclusion. Moreover, there are known examples of Killing horizons with vanishing surface gravity, thus it could be possible to proceed like in [PR18] to prove that a compact null hypersurface with vanishing surface gravity is a Killing horizon. If this is true, if [Haw92] is correct and if bounded horizons have vanishing surface gravity, we could conclude that compact Cauchy horizons are either unstable or symmetric.

Almost-closedness of incomplete generators

Let $\mathcal{H}$ be a compact Cauchy horizon. We say that a generator γ is *almost-closed* if for every $p \in \gamma$ and $t_0 \in \mathbb{R}$, $p \in \overline{\gamma([t_0, +\infty))}$.

Proposition D.3

Every generator of an incomplete horizon $\mathcal{H}$ is almost-closed.

Recall that even if $\mathcal{H}$ is not incomplete, it was shown in [Min19] there is at least one almost-closed generator.

Proof. The idea was introduced in [MI08] and consists in using the Poincaré reccurence theorem. Denote h the homogeneity vector field given by Theorem 5.1, and denote as usual $\omega = \omega_h$. As in [PR18], define a tensor $\tilde{g}$ of type $(0, 2)$ on $\mathcal{H}$ by

$$\tilde{g} := g + \omega \otimes \omega \ .$$

As $\omega(h) = -1$, and as g is positive-definite on the spacelike bundle $\ker \omega$, $\tilde{g}$ is a Riemannian metric on $\mathcal{H}$. Moreover, as $\kappa = -1$, $\mathcal{L}_h \omega = \mathrm{d}\kappa = 0$ by Lemma 5.6. This fact combined with Lemma 5.1 show that h is a Killing field for $\tilde{g}$. But then the flow $(\varphi^\rho)_{\rho \in \mathbb{R}}$ of h is a family of isometries of the compact Riemannian manifold $(\mathcal{H}, \tilde{g})$, thus preserve its Riemannian measure. Denote d the distance on $\mathcal{H}$ induced by $\tilde{g}$. For γ a generator and $p \in \gamma$, we will construct by induction with the Poincaré recurrence theorem a sequence $\rho_n \to +\infty$ such that $d(p, \varphi^{\rho_n}(p)) \leq 1/n$, which will prove that γ is almost-closed.

The initialisation of the induction for $n = 1$ is juste the Poincaré recurrence theorem applied to the dynamical system $(\varphi^k)_{k \geq 1}$. Suppose that $d(p, \varphi^{\rho_n}(p) \leq 1/n$. Then by the Poincaré reccurence theorem applied to the dynamical system $(\varphi^{k(\rho_n+1)})_{k \geq 1}$, there is an integer $k \geq 1$ such that $d(p, \varphi^{k(\rho_n+1)}(p)) \leq 1/(n+1)$, hence the induction defining $\rho_{n+1} = k(\rho_n + 1) \geq \rho_n + 1$. □

Denote $\mathcal{C}$ the chronology violating set of the spacetime $(\mathcal{M}, g)$, *i.e.* the set of points $p \in \mathcal{M}$ such that there is a closed timelike curve passing through p.

Corollary D.1

Let $\mathcal{H}$ be an incomplete compact Cauchy horizon. Then $\mathcal{H} \subseteq \overline{\mathcal{C}}$.

In other words, any incomplete compact Cauchy horizon is in the closure of the chronology violating set. This result highlights the relation ship between the loss of determinism and predictability that defines Cauchy horizons, and the violation of chronology.

For the detailed proof, see [GM21]. Basically, the idea of the proof is to follow the flow of an arbitrary small vector field defined on a generator γ, and to use Proposition D.3 to show that the created curve is almost-closed and timelike. It is then possible to close this curve as it passes through the past timelike cone of any of its point.

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